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LXXXIV. *Magnetic Energy and the Thermodynamics of Magnetization.*

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1. *Introduction.*

IN a paper written about two years ago [1], a number of thermodynamic relations for magnetization were derived. The basic equation used was equivalent to the following, referring to unit volume :

$$dE_1 = T\,dS + H\,dI. \quad . \quad . \quad . \quad . \quad . \quad (1.1)$$

Here S is the entropy and I the intensity of magnetization ; E_1 was referred to as the "internal energy." A related equation is

$$dE_2 = (E_1 - HI) = T\,dS - I\,dH, \quad . \quad . \quad (1.2)$$

in which E_2 was referred to as the "total energy, that is, the sum of the internal energy and the potential energy in the field of force." The relations which were primarily considered were those for the dependence of magnetization on field and temperature, in which the different types of dependence were shown to be related to different conditions for the variation of the internal energy with the magnetization at constant temperature,

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for the magneto-caloric effect, and for the effect of magnetization on specific heat. The purpose of the paper was partly to derive new relations and partly to bring out the interconnexion between the various relations, some of which were well known. The initial equations (1.1) and (1.2) were believed to have been established on a satisfactory basis (although the distinction between E_1 and E_2 had not been explicitly discussed), and a brief indication only was given of the manner in which they might be derived. Unfortunately, through a misguided attempt at simplification, the indication given in the case of (1.1) was completely unsatisfactory. This has been pointed out by Guggenheim, who, in two recent papers, has dealt with one side of the magnetic energy question in a very rigorous manner and has also considered certain aspects of the thermodynamics of magnetization.

In his first paper [2] Guggenheim derives, starting from Maxwell's equations, a general expression for the magnetic energy of a system of fixed configuration in the form

$$W_m = \frac{1}{4\pi}\int dV \int_0^B H\, dB, \quad . \quad . \quad . \quad . \quad (1.3)$$

and a similar expression (with E and D replacing H and B) for the electrostatic energy, W_e. He then shows that for any infinitesimal variation of the system, including variations of configuration, the increase δW of the energy is of the form

$$\delta W = \delta W_0 + \delta W_e + \delta W_m, \quad . \quad . \quad . \quad . \quad (1.4)$$

where W_0 depends on the configuration and state of motion of the system, but is independent of its electric and magnetic state. In the second paper [3] these results are made the basis of a thermodynamic treatment. For a system of fixed configuration, with $W_e = 0$, the total energy E and the free energy F are given by

$$E = E_0 + \frac{1}{4\pi}\int_{(S,\,V)} dV \int_0^B H\, dB, \quad . \quad . \quad . \quad (1.5)$$

$$F = F_0 + \frac{1}{4\pi}\int_{(T,\,V)} dV \int_0^B H\, dB, \quad . \quad . \quad . \quad (1.6)$$

where the integrations are at constant S, V for E and constant T, V for F, and E_0 and F_0 are independent of the

magnetic field. It is usually more convenient to deal with F than E ; the following expression for δF is obtained for a general variation at constant configuration :

$$\delta F = -S_0 \delta T + \frac{1}{4\pi}\int dV \int_0^B \frac{\partial H}{\partial T} dB \delta T + \frac{1}{4\pi}\int dV H \delta B. \quad \therefore \quad (1.7)$$

Expressions of a similar type are obtained for the other thermodynamic functions, and from them a number of thermodynamic relations for magnetization are derived by the usual method. (For full details and a treatment of topics not considered here, including an analysis of pressure volume effects, Guggenheim's papers should be consulted.)

The validity of the relations derived by Guggenheim in the way outlined is not in question. Many of them, however, appear to be somewhat inconvenient in form, and their connexion with other superficially simpler relations which have been widely applied is not immediately apparent. The diversity in form of the different types of expressions which have been given for magneto-thermal relations may, for the most part, be traced back to differences in the expressions used for magnetic energy ; and it is the purpose of this paper to examine the interconnexion between various energy expressions such as (1.3) and those implied by (1.1) and (1.2). This seems the more desirable in that some of Guggenheim's statements suggest considerable doubt as to the validity of other formulations which have been given, and generally accepted, for the thermodynamic relations.

Differences in expressions for magnetic energy may arise in three ways : firstly, through different choices of a standard state to which a zero energy is arbitrarily assigned ; secondly, through a difference in the method of specifying a state ; and, thirdly, through the use of the same term " magnetic energy " with different connotations. The first question is usually considered in some detail in standard treatises in connexion with the relation between elementary magnetic shells and current circuits ; and a good discussion of the question in connexion with (1.3) is given by Guggenheim. It need not be considered further here, particularly as it is generally irrelevant in connexion with derived relations, in which changes of energy are involved. It is the third type of difference with which this paper is primarily concerned, though it is often associated with differences

in specification. The kind of problem which requires clarification may be illustrated by a simple example. The hysteresis loss per unit volume of a ferromagnetic may be obtained (correctly) by integrating over a cycle any of the four expressions $\int H\,dB$, $-\int B\,dH$, $\int H\,dI$, and $-\int I\,dH$; but the precise significance of the different integrals over a non-cyclic path is by no means obvious. Usually no discussion of this point is given; by a happy —or unfortunate—accident the most dubious arguments may lead triumphantly to an undisputed conclusion for the hysteresis loss.

The main purpose of the paper is to show the relation between energy expressions involving integrals of the type $\int H\,dB$, $-\int I\,dH$, and $\int H\,dI$. The questions considered are all of an "elementary" kind, but an examination of a number of standard treatises and text-books certainly suggests that these particular elementary questions may be very confusing. A number of references were given in the previous paper [1]. To these should be added, as giving excellent treatments of magnetic energy, the books by Cohn[4] and Livens[5], and a paper by Larmor[6].

2. *The Magnetic Vectors* H, I, *and* B.

The primary magnetic characteristics of a material under defined conditions (at constant temperature and at constant volume, for example) may be specified by a set of associated pairs of values of H, I, and B. With the definitions ordinarily adopted the pairs chosen are usually H and I or H and B. The choice is in general made not on the ground that two of the three quantities H, B, and I are more fundamental than the third, but simply as a matter of convenience.

On the H, B system the characteristic material coefficient is the permeability, μ; on the H, I system, the volume susceptibility, κ. (A derivative of the H, I system is the H, σ system in which σ is the specific intensity of magnetization, that is, the magnetic moment per unit mass, with the mass susceptibility, χ, as the characteristic coefficient.) In technology the H, B system is almost exclusively used, the important relations being those which involve B directly rather than I. Moreover, the usual methods of magnetic testing (of the closed magnetic circuit type) lead directly to values of B corresponding to particular values of H. In connexion with

tests of the magnetometer type, and indeed with most of the ponderomotive methods which are used for determining the magnetic characteristics of non-ferromagnetics, the H, I formulation is more convenient. In the theoretical treatment of the magnetic properties of materials (except in immediate connexion with their use in technology) the H, I system is generally preferred.

While there can be little question of the appropriateness of using I in specifying material characteristics (if only because, in space free from matter, $I=0$), it might be argued that the choice of B, I would be as suitable as H, I. It is sometimes stated that B is more fundamental than H; in the text-book of Abraham and Becker [7], for example, is the statement: "Not the magnetic force H, but the induction B, is the primary magnitude. The vector $H=B-4\pi I$ . . . . must be regarded as purely artificial, only employed for greater convenience in the statement of the formulæ." As far as the formal molar theory is concerned, a distinction of this kind between B and H seems quite arbitrary. Such a distinction, however, appears to be made with reference to an atomic (or electronic) theory of the magnetic properties of matter in bulk.

In the usual form of electron theory there is only one type of magnetic vector, say h. The elementary magnet is an electron moving in an orbit, and the field arises from these orbital electrons (currents in molecular circuits) and from moving electrons corresponding to ordinary currents. It is clear without any detailed consideration that it is the macroscopic B (not H) which corresponds to the value of h averaged over a physically small volume. B thus corresponds to the total magnetic field, while H is simply the non-local part of the field. In a magnetized body the value of B over a physically small region is made up of a part (4πI) corresponding to the field of the molecular currents in that region and a part H due to the remaining parts of the magnetized body, the external magnetized bodies, and ordinary currents.

These questions of the relation between B and H are discussed very clearly and fully by Livens [5], who departs from the usual practice in nomenclature; he designates B not the magnetic induction, but the magnetic force; he still uses the term induction, however, in his designation

of H as " the magnetic induction (of mechanically effective force)." Although there is no logical objection to these designations, it seems unfortunate, in view of the very clear explanation given of the matter, that the issue should be confused by the use of a terminology which is the reverse of that which is customary. There seems little advantage, also, in using B in place of H for the magnetic force in free space, though this again involves no logical inconsistency. It may be noted, incidentally, that in some parts of his treatment Livens uses the I, B system; the corresponding material coefficient is $\mu'=I/B$. It should be remembered, in using Livens's formulæ, that the symbol μ is used not for the permeability but for $1-\mu'$. This is the inverse of the permeability, as " rational " units are used, with which $B=H+I$.

The relation considered above—that H corresponds to part of the total force B—is also obtained in a purely formal atomic treatment of magnetostatics, based on the consideration of magnetized bodies as aggregates of elementary magnetic particles. Such an artificial treatment, though sometimes convenient, leads to artificial difficulties if questions are asked about the interior of the particles. With the usual elementary current circuit treatment the divergence of B is zero. (This holds also in the magnetic particle treatment if the interior of the particles is regarded as inaccessible, so that the integrated normal induction over a closed surface is zero, it being understood that the surface can include only whole numbers of particles. Alternatively expressed, isolated poles are supposed never to occur.) It seems to be this solenoidal property of B which has suggested that B is more fundamental than H. From the point of view taken here, however, with any ordinary use of the term " fundamental " the question whether B or H is the more fundamental seems to be completely devoid of meaning.

In dealing theoretically with the magnetic properties of materials the primary question involved is how I depends upon the external field which can be applied by means of controllable magnets or currents external to the body (or portion of the body) under consideration. It is, then, generally convenient to consider the H, I set of relations rather than the B, I (or, for technological purposes, the H, B set). There is, however, a practical

complication, in that the field due to external magnets and currents, H_e, is not in general equal to the field H in the body ; H is made up of two parts, H_e, and a part due to the magnetization of the body itself. This part will be called H_i. The H, H_e, H_i relations will be further considered below.

3. *System of Magnetic Particles.*

There are, as is well known, two distinct types of formulation of expressions for magnetic energy. This may be illustrated most simply by considering a system of elementary permanent magnetic particles, which make up a number of finite magnetized bodies. By considering the work done in bringing these particles into position one by one, first in one order, and then in the reverse order, it may readily be shown [8] that the energy W of the system is given by

$$W = -\tfrac{1}{2}\Sigma_k \int H \cdot I \, dV_k, \quad . \quad . \quad . \quad . \quad . \quad (3.1)$$

where the integrations are extended over the magnetized bodies only. (The zero of energy is taken as the energy of the particles, with their associated fields, when they are scattered at infinity.) In the case considered there are no currents, so that

$$\int H \cdot B \, dV = \int H \cdot (H + 4\pi I) dV = 0, \quad . \quad . \quad (3.2)$$

the integrations being over all space. From these two equations, as an alternative expression for the energy,

$$W = + \int \frac{H^2}{8\pi} dV, \quad . \quad . \quad . \quad . \quad . \quad (3.3)$$

where the integration is over all space. Although (3.1) and (3.3) are equivalent, the two types of expression may differ widely in their convenience for particular applications.

A further expression for the energy may be obtained by supposing that each body is formed by aggregation from the particles at an " infinite " distance from all the other bodies (a distance, that is, at which the field due to the other bodies is effectively zero), and that then the bodies are brought into position. This process leads to the expression

$$W = -\tfrac{1}{2}\Sigma_k \int (H_i)_k \cdot I \, dV_k - \tfrac{1}{2}\Sigma_k \int (H_e)_k \cdot I \, dV_k, \quad . \quad (3.4)$$

where H_i is the field due to the rest of the same body at

an element, and H_e is the field due to all the other bodies. That (3.4) and (3.1) are equivalent is immediately obvious, since

$$H=H_e+H_i \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.5)$$

The second term in (3.4) is the energy corresponding to the attractive forces exerted by the bodies on each other, associated with the ponderomotive forces on the bodies as wholes; it may be called the external field energy, or in this case the mutual potential energy of all the bodies in the system. The first term is the energy corresponding to the attractive forces between the parts of the body; it may be called the demagnetizing field energy, since H_i is the demagnetizing field, which is determined by the distribution of I throughout the body. When I is uniform throughout the body $H_i=-DI$, where D is the demagnetization coefficient. (It seems better not to use the term "internal field energy" for the first term, as this is liable to lead to confusion with other uses of the term "internal.")

The energy W is the energy of the whole system of magnetized bodies, measured from the standard state. In many problems, however, it is not the energy of the whole system which is required, but the energy of one particular magnetized body in the field of the rest, that is the difference in energy of the system with and without the particular body in position. For the case under consideration, in which the magnetized bodies are regarded as made up of ultimate magnetic particles, this energy is readily obtained.

Let all the magnetic bodies except one, say k, be in position; then the additional work done in aggregating the magnetic particles to form the body k and to bring it into position is

$$W_k'=-\tfrac{1}{2}\int(H_i)_k\cdot I\,dV_k-\int I\cdot H_e\,dV_k. \quad . \quad . \quad (3.6)$$

W_k' represents the increase in the total magnetic energy of the whole system due to the addition of the body k. The total energy of the whole system is not, however, obtained by summing expressions similar to W_k' over all the bodies in the system; for the second term in the expression represents the mutual energy of the body k and the rest of the system, involving all the pairs of bodies of which one is k. In a summation this mutual energy would be counted twice. The total energy is the sum

of terms corresponding to the first term in (3.6), together with half the sum of those corresponding to the second, giving for the total energy W_k as in (3.4).

The system so far considered is particularly simple. The energy may be treated as potential in the simplest sense. Each body is regarded as made up of permanent magnetic particles. Their state of magnetization being independent of the field, so also is their internal energy. The energy associated with the particles (measured from the zero in which the particles are removed from the fields of other particles) is entirely external field energy, the external field for the particle being H_e+H_i. For bodies which are aggregates of magnetic particles, the associated energy may be regarded as made up of two parts, the demagnetizing field energy and the external field energy; both of these parts correspond to energy of the same kind—external field energy with reference to the particles. The internal or intrinsic energy of the particles, which has been taken as giving the arbitrary zero of energy, by hypothesis does not change. The essential and important distinction made here is that between the energy of such a body in a field (the change of energy of the whole system due to the presence of the body) represented by (3.6), and the contribution from each body, represented by (3.4), to the total energy of the system, regarded as a sum of contributions from integrals over the separate bodies. This distinction, though straightforward, may lead to a little confusion in the following way. Consider a system of $(n-1)$ bodies. The energy is represented by

$$W=-\tfrac{1}{2}\sum_1^{n-1}\int(H_i)_k\cdot I\,dV_k-\tfrac{1}{2}\sum_1^{n-i}\int(H_e)_k\cdot I\,dV_k. \quad (3.7)$$

The change of energy due to the addition of a body n is

$$W'_m=-\tfrac{1}{2}\int(H)_n\cdot I\,dV_n-\int(H_e)_n\cdot I\,dV_n. \quad (3.8)$$

The total energy of the system of n bodies is represented by

$$W=-\tfrac{1}{2}\sum_1^{n}\int(H_i)_k\cdot I\,dV_k-\tfrac{1}{2}\sum_1^{n}\int(H_e)_k\cdot I\,dV_k. \quad (3.9)$$

The appearance of the equations suggests that (3.7) and (3.8) do not add together to give (3.9). This, however, is due to the fact that the significance of $(H_e)_k$ in (3.7) is different from that in (3.9). Let $(H_k)_j$ be the field due to the body j at an element of the body k.

Then

In (3.7), $(H_e)_k=(H_1+H_2+\ldots+H_{k-1}+H_{k+1}+\ldots+H_{n-1})_k$.

In (3.9), $(H_e)_k=(H_1+H_2+\ldots+H_{k-1}+H_{k+1}+\ldots+H_{n-1}+H_n)_k$.

In each case $(H_e)_k$ is the field due to all the rest of the bodies in the system on elements of k, and it will change if the number of bodies (or their positions) are changed. If this is borne in mind, a brief consideration shows that (3.8) and (3.9) are compatible with each other.

The case of systems of magnetic particles aggregated into finite bodies has been dealt with here although it is somewhat artificial, because it helps to make clear some of the points which arise in the corresponding treatment of ordinary inductively magnetized bodies, particularly through the differentiation which has been introduced of external and demagnetizing field energy for finite bodies.

4. *Total Energy of an Inductively Magnetizable Body.*

The energy of inductively magnetizable bodies will now be considered. The simplest case is that of a single inductively magnetizable body (which will be specified by q) in a field produced by permanent magnets (specified by 1, 2 ... k ... n) of the kind considered in the last section. For the body q it will be supposed that I is a unique function of H under the conditions imposed, and that I=0 when H=0 (or $H_e=0$). With the permanent magnets fixed in position, the work done in bringing up the body q from a position at which H=0 is given by

$$W_q'=-\int dV_q\int_0^{H_e} I\cdot dH_e \quad . \quad . \quad . \quad . \quad (4.1)$$

This result, which is, of course, not new (*cf.* Cohn [4], p. 142), may be based on the consideration that the work done in moving an element of magnetic matter of moment m in an external field is

$$\delta W=-m\cdot\delta H_e \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.2)$$

This expression (4.2), which is, indeed, little more than an expression of basic definitions, would, presumably, be generally accepted. It has, however, been suggested by Guggenheim [2] that a formula of the type (4.2), though valid when the external field is "due entirely to fixed

rigid permanent magnets," " is not valid if there is any induced magnetism other than that in the element being moved," nor "for the displacement of a finite piece of magnetic matter with induced magnetism, because it ignores the change in the magnetization of one part of the matter moved induced by the change in the magnetization of the other parts." This might seem to imply that the extension (4.1) is incorrect, but it is possible that a difference in the use of terms is involved. It is necessary, therefore, to be quite explicit about the significance of W_q' and H_e in (4.1).

As in the last section, H_e in (4.1) is the external field, that is the field due to all bodies external to V_b (in this case the permanent magnets 1, 2 . . . . n), and does not include the demagnetizing field H_i due to the magnetization of other parts of V_q. Correspondingly, W_q' is the work done on the system by " external " agency. In the present case, it may be noted, it is work done in the most primitive sense, which could be measured by a rope attached to V_q, a weight and a pulley.

The changes which V_q may undergo as it is magnetized (such as changes of volume or shape, or changes of temperature) are irrelevant to the validity of (4.1); though the functional dependence of I on H will, of course, depend on the further conditions which may be imposed (such as that the volume and temperature should be constant). W_q' is simply the external work done on the system; but the form taken by the added energy is a matter for further analysis.

If there are other inductively magnetizable bodies (as well as, or instead of, the permanent magnets) in the system in its initial state, (4.1) still holds good, the value of H_e still being the field due to all the other bodies in the system. The value of H_e in the region finally occupied by V_q, will, however, in this case differ from the initial value; and the actual calculation of the integral in (4.1) might well present prohibitive difficulties.

If the external field is produced in part or wholly by currents, the processes which occur on bringing up a magnetizable body may be more complex. Provided that the currents are the same in the final as in the initial state, however, (4.1) still represents the total work done on the system. If the currents are maintained constant throughout the whole course of the change, the conditions correspond to those with permanent

magnets, and W_q' is the increase of energy of the whole system which is equal to the mechanical work done on the system. If the current circuits are supplied by an external battery, then, under the usual conditions, there will be an additional E.M.F. round the circuit due to the motion of the magnetized body, and while V_q is moving the current will differ from that in the initial and final states. The mechanical forces will therefore differ from those under constant current conditions, but the increase in the mechanical work done on the system will be exactly balanced by the increase in external work done by the system through the operation of the additional E.M.F. (The additional energy may be used, for example, in charging up external batteries, or, in principle, through the intermediary of a motor, to raise weights.) It is unnecessary to give a detailed proof of this, as it may proceed on precisely the same lines as Guggenheim's treatment of the energy balance in connexion with the $\int H\,dB$ expression (Guggenheim [2]). Thus W_q' represents the excess of the work done on the whole system over that done by it; in the case of systems of magnetized bodies only (with no currents) this work may conveniently be purely mechanical; when there are currents, it may be inconvenient to change the energy of the system directly by mechanical methods, though indirectly the whole energy change may be produced by, and expressed in terms of, mechanical work done on the system.

It is concluded that W_q' in (4.1) represents the total work done on a system containing any or all of permanent and induced magnets and currents due to moving into it an inductively magnetizable body V_q, provided that no bodies other than V_q are moved in the process (and no external work is done on or by them) and the currents are the same in the final as in the initial state. If the whole system is thermally isolated, W_q' represents the total change of energy of the whole system in the change considered. Since W_q' represents the difference in energy of the system with and without V_q, W_q' may also be referred to as the total (magnetic) energy of an inductively magnetizable body in an external field (as measured from the state when $H=0$, which is taken as the zero). With W' used in this sense, it is legitimate to write

$$\delta W_q' = -\int dV_q I \cdot \delta H_e. \quad . \quad . \quad . \quad . \quad (4.3)$$

If, for example, with V_q fixed, the external field is changed from H_1 to H_2,

$$(W_q')-(W_q')_1=-\int dV_q\int_{H_1}^{H_2} I\cdot dH_e. \quad . \quad . \quad (4.4)$$

The change in external field may be brought about by changing currents, or by moving permanent magnets or other inductively magnetizable bodies; the change of energy represented by (4.3) will not necessarily, or even generally, represent the change in energy of the entire system concerned, but merely the change in total energy associated with V_q.

With regard to the total energy of the whole system, the position is similar to that considered in the last section. The total energy of a system of magnets and magnetizable bodies is not obtained by summing expressions of the type (4.1) over all the bodies in the system. For many purposes expressions for the total (magnetic) energy of the whole system are unnecessary; they are, however, required in order to compare the results of the present treatment with those obtained by other methods; and for this a further analysis of the total energy of a body in a field is required.

5. *Energy of a System.*

It will be convenient to consider again, at first, a single inductively magnetizable body q, in a field due to external magnetized bodies. The total energy of V_q as given in the last section may be treated as the sum of two parts:

$$W'_q=-\int dV_q\int I\cdot dH_e \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.1)$$

$$=-\int dV_q I\cdot H_e+\int dV_q\int H_e I dI. \quad . \quad (5.2)$$

The first term in this expression is the external field energy, or, in this case, the mutual potential energy of V_q with all the rest of the magnetized bodies in the system. This term is the same as that in (3.6); its magnitude is independent of whether the magnetization is permanent or induced, or of the manner in which I has varied to attain its final value. It corresponds to the work required to remove V_q to a region in which $H=0$, keeping the value of I unchanged. It may be noticed that the bodily force on V_q as a whole is given by

$$F_q=\int dV_q \operatorname{grad}_H (IH_e), \quad . \quad . \quad . \quad (5.3)$$

the operation grad being applied only to H_e. The energy represented by the second term includes demagnetizing field energy and any further internal magnetic energy of V_q, again measured from the state in which $H_e=0$.

The total energy of a system of magnetized bodies is now readily obtained. The external field energy is obtained as in dealing with magnetic particles, and to this must be added the sums of the internal energies

$$W_{System}=-\tfrac{1}{2}\Sigma_k\int dV_k I\cdot H_e+\Sigma_k\int dV_k\int H_e\,dI. \quad . \quad (5.4)$$

The integrations are over the volumes occupied by the magnetized bodies (including permanent magnets). The first term in (5.2) is the sum of the mutual energies of pairs of magnetized bodies, each pair including V_q.

If there are currents in the system, there is additional energy which need not be considered here in detail. This energy may be expressed as a sum of integrals over surfaces bounded by the current circuits, or, by a well-known transformation, as $(1/8\pi)\int H\cdot B\,dV$ taken over the whole of space. (In the absence of ordinary currents this integral is zero.) The total excess energy of the system above that in the state in which $H_e=0$ and in which the currents are zero is thus given by

$$W_m=\frac{1}{8\pi}\int H\cdot B\,dV-\tfrac{1}{2}\Sigma_k\int dV_k I\cdot H_e+\Sigma_k\int dV_k\int H_e\cdot dI. \quad . \quad . \quad . \quad . \quad (5.5)$$

The manner of derivation shows that this expression is appropriate for a system in which the positions of the finite bodies are regarded as fixed. The volumes and shapes of these bodies when the external field is H_e may differ from those when the external field is zero.

It is now desirable to show that this expression is in accordance with (1.3), obtained by the use of Maxwell's equations. The value (1.3) will be denoted by W_m', and it must be remembered that this expression for the energy applies to a system of fixed configuration, as it is the excess of the energy over that of the system with the same configuration for $H_e=0$.

$$W_m'=\frac{1}{4\pi}\int dV\int H dB=\frac{1}{8\pi}\int H^2\,dV+\int dV\int H dI. \quad (5.6)$$

Now

$$H^2=H\cdot(B-4\pi I). \quad . \quad . \quad . \quad . \quad . \quad (5.7)$$

Substituting,

$$W_m'=\frac{1}{8\pi}\int H\cdot B\,dV-\tfrac{1}{2}\int H\cdot I\,dV+\int dV\int H\cdot dI. \quad (5.8)$$

The second and third of these integrals extend effectively only over magnetized bodies.

$$-\tfrac{1}{2}\int H\cdot I\,dV=-\tfrac{1}{2}\sum_k\int dV_k I\cdot H_e-\tfrac{1}{2}\sum_k\int dV_k I\cdot H_i, \quad . \quad (5.9)$$

$$\int dV\int H\,dI=\sum_k\int dV_k\int H_e\cdot dI+\sum_k\int dV_k\int H_i\cdot dI. \quad (5.10)$$

Substituting these values in (5.8), and referring to (5.5),

$$W_m'-W_m=\tfrac{1}{2}\sum\int dV_k\int(H_i\cdot dI-I\cdot dH_i). \quad . \quad (5.11)$$

The difference $W_m'-W_m$ arises simply from the difference in the standard state to which the energy is referred; W_m' applies to a system of fixed configuration, implying that the volumes and shapes (the disposition of the material elements) of the finite bodies are the same in the standard zero state as in the state considered. If the same standard state is chosen for W_m, the integral in (5.11) reduces to zero, since, for a rigid body,

$$\int dV\int(H_i\cdot dI-I\cdot dH_i)=0. \quad . \quad . \quad . \quad (5.12)$$

To show this simply, let m_f, m_g be the magnetic moments of two typical volume elements of the body; H_i at f is the sum of the fields due to all other elements, and the field due to g at f may be represented by $m_g f(r_{gf})$, it being unnecessary to specify the form of f more completely.

$$\int dV\int H_i\cdot dI=\sum_f\int\{\sum_g m_g f(r_{gf})\}\,dm_f, \quad . \quad (5.13)$$

$$\int dV\int I\cdot dH_i=\sum_f\int m_f d\{\Sigma m_g f(r_{gf})\}. \quad . \quad . \quad (5.14)$$

The integrals are clearly equal to each other if the r_{gf} are invariant.

It should be noticed that, although the integral in (5.11) differs from zero only when there are deformations, it is not, by itself, an expression for the magneto-deformation energy; for the remaining integrals, such as $\int H_e\cdot dI$, derived from the expressions for W_m and W_m' can only be identified if they refer to changes of I

with H_e under the same conditions. In the method followed for deriving an expression for W_m it is not necessary to take deformation explicitly into account; the work done by the external field in deforming the body is included in the general expression (5.1). In comparing the expression for the total energy of a system derived from (5.1), however, with an expression (5.6) which is valid for the magnetic energy when the integration is at fixed configuration, the "fixed configuration" condition must be applied throughout.

In this section it has thus been shown that the expression (5.1) for the external work done on a magnetizable body when the external field changes from 0 to H_e, is compatible with the general expression (5.6) for the magnetic energy of a complete system.

For some types of problem an expression for magnetic energy of a complete system in the form of an integral over the whole of space is of considerable value; it does not, however, seem immediately suitable for dealing with the thermodynamic relations of particular magnetized bodies. To find the difference of energy of a given body V_q in two different states corresponding, say, to different values of B, H throughout the body it is not sufficient to integrate $1/4\pi\int H\cdot dB$ over the volume V_q for the two states; it is necessary to find the difference in $(1/4\pi)\int dV\int H\cdot dB$ over the whole of space for the two states of V_q. For the variation corresponding to bringing up an initially unmagnetized rigid body V_q into a field it may be shown that

$$\delta(1/4\pi)\int dV\int H\cdot dB = -\int dV_q\int I\cdot dH_e\,;$$

in this and other cases it is simply a matter of convenience to make direct use of expressions in the form of integrals over V_q only, which can be derived in the manner indicated above.

6. *Internal Energy, Demagnetizing Field Energy, and External Field Energy.*

Let E_T be the total energy per unit volume of a magnetizable body, of volume V. From the expression for the work done in a change of external field by δH_e,

$$\int dV\,\delta E_1 = \int dV\,T\delta S - \int dV\,I\cdot\delta H_e, \quad . \quad . \quad (6.1)$$

the integration being over the volume of the body. This expression will be valid either at constant volume, or in the absence of applied forces ; though the dependence of I on H would not strictly be the same in the two cases. If there are applied forces, a further term of the type $-\int dV\mathrm{P}\delta\mathrm{V}$ must be added, where P is the applied pressure.

For changes in which there is no deformation of the body, which may be referred to, for brevity, as changes at constant volume, it is permissible to write

$$d\mathrm{E}_\mathrm{T} = \mathrm{T}\,d\mathrm{S} - \mathrm{I}\cdot d\mathrm{H}_e. \quad . \quad . \quad . \quad . \quad (6.2)$$

Although (6.1) is valid even when there are changes of volume accompanying magnetization, this does not hold, in general, for (6.2) ; for these changes of volume result in changes of internal (including deformation) energy which may not vanish for volume elements, although the sum is zero for the whole body. The equation (6.2) may, however, be taken as a very close approximation even for changes at constant (zero) pressure in most practical applications. The limitations to its generality, though of formal interest, are of little practical importance.

The main difficulty in connexion with (6.2) from a thermodynamic standpoint arises from the fact that, for a given material, I is a unique function (say for S or T and V constant) not of H_e but of H. Even if the dependence of I on H is known, for a given value of H_e, it is not possible in the general case to calculate the distribution of H (or I) throughout a body in the field. Moreover, no simple specification of the state of the body can be given. The same difficulty arises in the experimental determination of the dependence of I on H. The difficulty is overcome by using bodies of such geometrical form that I is uniform throughout the body when it is placed in a uniform external field. A simple specification of the state of the body is then possible. For such a body

$$\mathrm{H}_i = -\mathrm{DI}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.3)$$

where D is the demagnetization coefficient. As is well known D can be calculated for ellipsoids, including the special cases of the sphere ($\mathrm{D}=4\pi/3$) and the (infinite) disk ($\mathrm{D}=4\pi$) ; for a long rod, D approaches zero, though I is not strictly uniform, except for an elongated ellipsoid.

For a ring placed within a uniformly wound solenoid (as in ring tests of iron) the state of uniformity of I with D=0 may be closely approached; this is the practicable counterpart of an endless cylinder in a uniform longitudinal field.

The part of the energy change corresponding to demagnetizing field energy may conveniently be separated.

$$d\mathrm{E_T}=\mathrm{T}d\mathrm{S}-\mathrm{I}\cdot d\mathrm{H}+\mathrm{I}\cdot d\mathrm{H}_i. \quad . \quad . \quad . \quad (6.4)$$

Let $\mathrm{E_D}$ be the demagnetizing field energy per unit volume. For changes at constant volume and with close approximation at constant pressure,

$$\mathrm{E_D}=-\int \mathrm{I}\cdot d\mathrm{H}_i=-\int \mathrm{H}_i\cdot d\mathrm{I}=\tfrac{1}{2}\mathrm{D}\mathrm{I}^2=\tfrac{1}{2}\mathrm{H}_i^2/\mathrm{D}. \quad . \quad (6.5)$$

(D is itself a function of shape; and equating the two integrals, in the zero applied force case, is equivalent to neglecting the effect of the change in D resulting from magnetization deformation.) The equation (6.4) may be rewritten

$$d\mathrm{E_T}=\mathrm{T}\,d\mathrm{S}-\mathrm{I}\cdot d\mathrm{H}-d\mathrm{E_D}. \quad . \quad . \quad . \quad . \quad (6.6)$$

Now the total energy may be regarded as a sum of the external field energy ($\mathrm{E_E}$), the demagnetizing field energy ($\mathrm{E_D}$), and the internal energy ($\mathrm{E_I}$).

$$\left.\begin{aligned} d\mathrm{E_T}&=d\mathrm{E_I}+d\mathrm{E_E}+d\mathrm{E_D},\\ &=d\mathrm{E_I}+d(-\mathrm{I}.\mathrm{H}_e)-\mathrm{H}_i\cdot d\mathrm{I},\end{aligned}\right\} \quad . \quad . \quad (6.7)$$

giving

$$\left.\begin{aligned} d\mathrm{E_I}&=\mathrm{T}\,d\mathrm{S}-\mathrm{I}\cdot d\mathrm{H}_e+\mathrm{I}\cdot d\mathrm{H}_e+\mathrm{H}\cdot d\mathrm{I}\,.\\ &=\mathrm{T}\,d\mathrm{S}+\mathrm{H}\cdot d\mathrm{I}.\end{aligned}\right\} \quad . \quad . \quad (6.8)$$

This expression, again, is valid at constant volume; it is approximately valid in the absence of applied pressures and tensions, neglecting then only changes resulting from deformation under the action of the field.

It should be noticed that this is a neglect not of deformation energy, but of what may be called non-uniformity of distribution of deformation energy through the body as a whole. Integrated over the whole body, (6.8) would have the same range of validity as (6.1). Actually, the dependence of I on H at zero external pressure will differ little from that at constant volume; this follows from the fact that the deformation energy is of the order $\frac{1}{2}k(\Delta\mathrm{V}/\mathrm{V})^2$ per unit volume, where k is

the bulk modulus and ΔV the change of volume. With ΔV as the value corresponding to a change of the intensity of magnetization from 0 to I, this is ordinarily very small compared with $H \cdot dI$; in unfavourable cases (for ferromagnetics) the ratio may be of the order 10^{-4}, but it is usually much less [1] (p. 578). The detailed consideration of magneto-deformation questions is thus hardly relevant to the main subject-matter of this paper, although in other connexions they are of considerable importance. The theory of magneto-deformation has been worked out in detail by Kapitza for non-ferromagnetics and by Gans and others for ferromagnetics. (A summary of this work and references are given in 'Magnetism and Matter' [9].)

The two energies E_E and E_D are of the same kind; it is the two together which correspond to external field energy for the elements of volume. Any thermal energy, or energy of molecular, atomic, or electronic motion is included in E_I.

The usual method of deriving the $H\,dI$ term in (6.8), which has been criticized, is to proceed from the Maxwell equation expression for the total energy of a system as follows :—

$$\frac{1}{4\pi}\int dV \int H \cdot dB = \frac{1}{4\pi}\int dV \int H \cdot d(H + 4\pi I)$$

$$= \int dV \frac{H^2}{8\pi} + \int dV \int H \cdot dI. \quad . \quad (6.9)$$

The first term corresponds to the total external field energy, associated with the attractive forces between magnetic elements and current circuits; for a system of magnetic elements it depends only on the intensity of magnetization and the configuration, being the same for a system of inductively magnetized elements as for idealized permanently magnetized elements. It is a matter of convenience to consider the second term only in questions of the internal energy of magnetized bodies. In the light of the above treatment this separation seems quite legitimate. It does not involve an arbitrary differentation of "energy of the ether" and "energy of the matter." The first term must, of course, be considered also in dealing with such problems as the equilibrium state of magnetization of a body in an external

field. The whole treatment might be developed by starting from the $\int \mathrm{H}\,d\mathrm{B}$ expression, but a different approach has been adopted here, which enables the significance of particular expressions to be more readily appreciated. The procedure would be to make suitable transformations and to identify the terms. As illustrative,

$$\frac{1}{4\pi}\int dV \int \mathrm{H}\cdot d\mathrm{B} = \frac{1}{8\pi}\int dV\,\mathrm{H}\cdot\mathrm{B} + \int dV \int \mathrm{H}\cdot d\mathrm{I} - \tfrac{1}{2}\int dV\,\mathrm{I}\cdot\mathrm{H}. \quad . \quad . \quad . \quad (6.10)$$

In this transformation the first term corresponds to the energy of currents, the second to the internal energy of magnetized bodies, the third to the total external field energy of magnetized bodies. The subdivision of this third term into demagnetizing field energy ($-\frac{1}{2}\int dV\,\mathrm{I}\cdot\mathrm{H}_i$) and external field energy ($-\frac{1}{2}\int dV\mathrm{I}\cdot\mathrm{H}_e$) has been sufficiently discussed, and also the replacement of the factor $\frac{1}{2}$ by 1 in considering the external field energy of a single magnetized body in an external field.

7. *Thermodynamic Functions.*

The equation for the internal energy

$$d\mathrm{E}_\mathrm{I} = \mathrm{T}\,d\mathrm{S} + \mathrm{H}\cdot d\mathrm{I} \quad . \quad . \quad . \quad . \quad . \quad (7.1)$$

corresponds to (1.1), the equation used previously [1]. In the absence of applied forces, and neglecting the effect of magneto-deformation changes, the state is uniquely defined by T or S and H or I. The equation gives the relation

$$(\partial\mathrm{T}/\partial\mathrm{I})_\mathrm{S} = (\partial\mathrm{H}/\partial\mathrm{S})_\mathrm{I}, \quad . \quad . \quad . \quad . \quad (7.2)$$

and three similar relations of the Maxwell type may be derived by considering the corresponding equations for the functions $\mathrm{E}_\mathrm{I}-\mathrm{TS}$, $\mathrm{E}_\mathrm{I}-\mathrm{HI}$, $\mathrm{E}_\mathrm{I}-\mathrm{TS}-\mathrm{HI}$. For a discussion of these relations, and an account of conclusions which may be derived by making use of them, reference may be made to the earlier paper [1].

The equation for the total energy per unit volume of a magnetized body (6.4 and 6.7), namely,

$$d\mathrm{E}_\mathrm{T} = \mathrm{T}\,d\mathrm{S} - \mathrm{I}\cdot d\mathrm{H}_e = \mathrm{T}\,d\mathrm{S} - \mathrm{I}\cdot d\mathrm{H} + \mathrm{I}\cdot d\mathrm{H}_i, \quad . \quad (7.3)$$

is awkward in that it involves implicitly the shape of the whole body. It may be written

$$d(E_T+E_D)=T\,dS-I\cdot dH \quad . \quad . \quad . \quad . \quad (7.4)$$

with the reservations discussed above. For dia- and paramagnetics the demagnetizing field term may usually be neglected; for ferromagnetics it may be made effectively zero for the special (but practical) case of a ring with the field produced by uniform winding of the ring. In these cases, with close approximation,

$$dE_T=T\,dS-I\cdot dH. \quad . \quad . \quad . \quad . \quad (7.5)$$

This equation is sometimes made the starting-point of a thermodynamic treatment (for example, by Bloch[10]), a procedure which is quite justifiable provided that its range of validity is borne in mind. The equation corresponds to (1.2), as given previously. An equation of this form can be derived at once from (7.1), and it leads to precisely the same set of "Maxwell relations." For a body in which I is uniform in a uniform field H_e the state may be specified by H_e instead of H; the equation (7.3) then leads to a set of Maxwell relations in which H_e replaces the previous H. As examples,

$$(\partial T/\partial I)_S=(\partial H_e/\partial S)_I, \quad . \quad . \quad . \quad . \quad (7.6)$$

$$(\partial T/\partial H_e)_S=-(\partial I/\partial S)_{H_e}. \quad . \quad . \quad . \quad (7.7)$$

The equivalence of (7.6) and (7.2) implies $(\partial H_i/\partial S)_I=0$, which is valid in so far as it is justifiable to put $H_i=-DI$ with D constant. The equation (7.7) may be written, with C_{H_e} as the specific heat in constant external field,

$$(\partial T/\partial H_e)_S=-(T/C_{H_e})(\partial I/\partial T)_{H_e}, \quad . \quad . \quad (7.8)$$

in which the magneto-caloric relations are expressed in terms of quantities which may be directly determined from experiment. This equation would be appropriate if the aim were to find the change of temperature of a particular body due to a change of external field; in the theoretical treatment of internal energy effects[1, 11], however, it is $(\partial T/\partial H)_S$ which must be used—derived, of course, by correction of the observed $(\partial T/\partial H_e)_S$.

The essential distinction between the two types of relation is that terms involving H are characteristic of the material, and those involving H_e may differ, for the same material, according to the shape of the body.

The thermodynamic functions which have the widest range of application are those based on E_T. Thus the free energy of a body in an external field is obtained by integrating F_T over the body, F_T satisfying the equation

$$dF_T = -S\,dT - I\cdot d\mathbf{H}_e + \Sigma X dx, \quad . \quad . \quad (7.9)$$

where X represents a generalized applied force. (For a uniform hydrostatic pressure this term would give $-P\,dV$, where P is the inward pressure.)

It is of interest to consider a case in which the distinction between H and H_e is significant other than for ferromagnetics. Let $I=\kappa H$; then, at constant T,

$$F = F_0 - \int \kappa H\cdot d\mathbf{H}_e. \quad . \quad . \quad . \quad . \quad (7.10)$$

If H_e is uniform and the shape of the body is such that

$$H_i = H - H_e = -DI,$$

$$F = F_0 - \frac{\kappa H_e^2}{1+D\kappa}. \quad . \quad . \quad . \quad . \quad (7.11)$$

Whether κ is positive or negative, F is a minimum for D a minimum. The equilibrium position for an elongated isotropic body in a uniform field is thus along the field, that is, axially between large pole pieces, whether the body is dia- or paramagnetic; contrasting with the usual equatorial and axial setting corresponding to the states of minimum energy when the field is not strictly uniform.

The total free energy may be sub-divided in a similar way to the total energy in (6.7). When that sub-division is adopted it may be noted that the term $-I\cdot H_e$ (or $-I\cdot H$ when the demagnetizing field energy may be neglected) is sometimes referred to as the "external free energy."

The validity of the main applications which have been made in the first paper [1] and elsewhere [11] of thermodynamical arguments appears to be confirmed by the present treatment, and it is unnecessary to consider them further. A number of refinements might be made, but it is believed that the bearing of these on the main points which were under consideration is hardly sufficiently important to call for more detailed treatment.

One of the questions requiring further consideration is that of magnetic hysteresis effects, and it is hoped to discuss these more fully in a later paper. It may,

however, be mentioned that from the point of view developed here the hysteresis loss formula may be simply derived, and in such a way that the significance of the integrals over non-closed paths is readily apparent, by considering the work done on the body. Let W_h be the work done on a ferromagnetic per unit volume by the external field during a cycle, Q_h the heat absorbed. Let $\overline{Q}_h = -Q_h$ be the heat evolved. In the following equations the integrals are all to be taken over a complete cycle.

$$\delta W = -I \cdot \delta H_e,$$

$$W_h = -\int I \cdot dH_e,$$

$$\delta E_T = \delta Q - I \cdot \delta H_e,$$

$$\int dE_T = 0 = Q_h - \int I d \cdot H_e, \quad . \quad . \quad . \quad (7.12)$$

$$Q_h = -Q_h = W_h = -\int I \cdot dH_e. \quad . \quad . \quad (7.13)$$

For a non-closed path $\delta\overline{Q}$ corresponds to the excess of the work done on the body by the external field over the increase in total energy. The result (7.12) will be generally valid when used in integrated form for the whole body. For a complete cycle, a number of alternative forms may be used :

$$\overline{Q}_h = -\int I \cdot dH_e = -\int I \cdot dH = +\int H \cdot dI$$

$$= \frac{1}{4\pi}\int H \cdot dB = -\frac{1}{4\pi}\int B \cdot dH. \quad (7.14)$$

This follows from the fact that all the integrands can be expressed in the form

$$d\phi = -I \cdot dH_e + dA, \quad . \quad . \quad . \quad . \quad . \quad (7.15)$$

where A is a unique function of the state. The questions requiring further consideration are those connected with the differentiation of the reversible and irreversible effects ; and, a question which goes outside thermodynamics, the interpretation of these in terms of the processes occurring in the material.

My previous paper on this subject led to a long correspondence between Mr. E. A. Guggenheim and myself. It is a pleasure to acknowledge the stimulus derived from his penetrating comment and criticism.

Summary.

In view of the numerous apparent inconsistencies in the treatment of magnetic energy, the question is reconsidered. The problem is approached in a direct manner by considering the mechanical work done in moving elementary magnetic particles, bodies consisting of aggregates of such particles, and inductively magnetizable bodies in an external field; first the field produced by a system of fixed rigid permanent magnets and finally in any way whatever.

The following conclusions are reached as strictly applicable to rigid bodies. For a single inductively magnetizable body, the magnetic contribution to the total energy of the body in an external field H_e is $-\int_0^{H_e} I \cdot dH_e$ per unit volume, the integration being over the volume of the body. This may be regarded as made up of contributions to the external field energy, the demagnetizing field energy, and the internal energy, these being respectively,

$$-\int d(I \cdot H_e), \quad -\tfrac{1}{2}\int d(I \cdot H_i), \quad \text{and} \quad +\int H \cdot dI$$

per unit volume, with the relation

$$H = H_e + H_i.$$

A set of thermodynamic functions may be based on the total energy function, E_T per unit volume, satisfying

$$\delta E_T = T\delta S - I \cdot \delta H_e,$$

and a second set on that for the internal energy, E_I per unit volume, satisfying

$$\delta E_I = T\delta S + H \cdot \delta I.$$

The relations derivable on the basis of the second of these equations have previously been developed. In most cases the demagnetizing field H_i is negligible compared with H_e, and the first equation leads to the same relations as the second. A brief discussion is given of cases where the distinction between H and H_i may not be neglected.

It is shown that the above results are compatible with the expression $(1/4\pi)\int dV \int H \cdot dB$ integrated over the

whole of space at fixed configuration for the magnetic energy of a complete system. This expression may also be used in forming thermodynamic functions, from which valid relations may be derived ; but these functions and relations appear not to be so immediately convenient for the actual applications which are ordinarily required.

When the "rigidity" restriction is removed, account must be taken of work done against or by externally applied forces, and of magneto-deformation effects. There is no particular difficulty in connexion with the first, but the second leads to a problem which, in the general case, is of great complexity. It is concluded, however, that the error made in using relations of the type

$$d\mathrm{E}_{\mathrm{I}}=\mathrm{T}\,d\mathrm{S}+\mathrm{H}\cdot d\mathrm{I},$$

when there are no applied forces, is negligible.

In view of the statements sometimes made as to the relative " fundamentality " of B and H, a brief discussion is given. Specifications of magnetic characteristics, and of the magnetic state of a system, may be made in terms of any pair of the three quantities I, B, and H ; and it is suggested that the choice is made simply on grounds of convenience for particular purposes.

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LXXXV. *Dielectric Constant of Ionized Gases.* *By* Ali Imam, *M.Sc.*, *and* S. R. Khastgir, *D.Sc.* (*Edin.*), *Reader in Physics, Dacca University* *.

1. *Introduction.*

DURING recent years a large number of investigations† has been carried out to measure the dielectric constant of an ionized medium in order to test the validity of the Eccles-Larmor theory. Most of the investigations are of a qualitative nature. Recently Mitra and Banerjee ‡ have given some experimental results with ionized air in the region of ultra-short wave-lengths, which show quantitative agreement with the prediction of the Eccles-Larmor theory. Prasad and Verma § have also published some experimental results with an electron atmosphere in a screen-grid valve, using waves of medium wave-lengths (81 to 520 m.).

In the present paper are given the method and the results of some of our measurements of the dielectric constants of a few ionized gases for different ionic concentrations and for a certain range of ultra-short wave-lengths. Briefly the method is to set up an ultra-short wave oscillatory circuit with a suitable coil and a capacity. The capacity is that of a condenser of two cylindrical metal pieces fixed tightly against the outside surface of a discharge-tube. A Lecher wire system inductively coupled to the coil of the oscillating circuit measures accurately the change in the wave-length of the oscillation when the medium intervening the two electrodes is ionized by passing a discharge. Two distinct sets of experiments have been performed :—

(1) Keeping the wave-length of the valve-oscillator constant, the discharge current has been varied and the

* Communicated by the Authors.

† Barton and Kilby, Phil. Mag. (6) xxvl. p. 567 (1913) ; Van der Pol, ' Thesis,' Utrecht Univ. (1920) ; Gutton and Ciement, *Com. Rendus*, clxxxiv. p. 441 (1927) ; Gutton, *Ann. de Phys.* xiii. p. 62 (1930) ; Szekely, *Ann. der Phys.* (5) p. 112 (1929) ; Appleton and Childs, Phil. Mag. (7) x. p. 969 (1930) ; Appleton and Chapman, Proc. Phys. Soc. xliv. p. 246 (1932) ; Benham, Phil. Mag. (7) xi. p. 457 (1931) ; Bergmann and Düring, *Ann. der Phys.* (3) i. p. 1041 (1929) ; Sil, Phil. Mag. xvi. p. 1114 (1933).

‡ Mitra and Banerjee, ' Nature,' Sept. 28, 1935.

§ Prasad and Verma, *Zeits. f. Phys.* xcix. pp. 7–8 (1936).

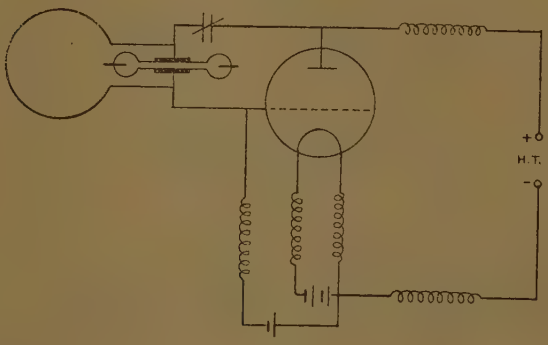
shift of the tuning-point of the Lecher system accurately measured. Thus for different ionic concentrations the values of the dielectric constant have been estimated from these measured shifts.

(2) Keeping the tube current constant, the wave-length of the valve-oscillator has been varied and a series of wave-length measurements made with the discharge current on and off.

2. *Experimental Arrangements.*

The circuit arrangement adopted in these measurements is shown in fig. 1. The oscillatory circuit consists

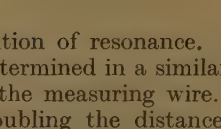
Fig. 1.



of a single turn 2 to 3 inches in diameter of 1/8 inch copper tube, and a condenser of two cylindrical brass plates fixed tightly against the glass side of a discharge-tube about $\frac{1}{4}$-inch in diameter. A small variable condenser has been employed for retro-action, while suitable choke coils having a self-resonance at approximately the working wave-length have been used in all the direct current supply leads to the valve electrodes. A Philips T.C.03/5 valve has been employed. A suitable grid-bias voltage has been found effective in producing oscillations of sufficient energy. The single turn of the copper tube has been coupled loosely to another loop, the ends of which are connected to the ends of a pair of parallel Lecher wires. When there is no discharge through the tube, the wave-length of the valve-oscillator is measured

by the Lecher wire system. For accuracy of measurements a third parallel wire fitted with a long wooden scale running parallel to it has been fixed close to one of the two Lecher wires. A copper bridge mounted on a suitable ebonite handle can be moved across the pair of Lecher wires, and the position of the bridge noted for resonance by observing the quenching of a small lamp in the oscillatory circuit. A telescope has been used for the observations whenever necessary. An A.C. milliammeter has also been used for determining the positions of resonance. The milliammeter has been suspended from the Lecher wires by two vertical rods of the same size, the upper ends of which slide along the wires and the lower ends are rigidly fixed to the terminals of the ammeter. The milliammeter has been allowed to rest on a suitable stand. A slider with a pointer is then fixed on the third measuring wire and the scale reading

Fig. 2.



noted for the position of resonance. The next position for resonance is determined in a similar way and fixed by another slider on the measuring wire. The wave-length is obtained by doubling the distance between the two positions. When a discharge is passed through the tube there is a change in the capacity of the two metal plates fixed against the outside surface of the discharge-tube. This change in the capacity is indicated by a change in the wave-length of the oscillation. The modified wave-length is then measured in a similar way.

The two cylindrical brass plates are mounted as shown in fig. 2. Semicircular cylindrical grooves, one in each of the two ebonite plates, make a round passage for the discharge-tube when the plates are pressed one against the other. The cylindrical brass pieces are inserted, one in each groove, and the ebonite plates are tightly screwed. This condenser, in parallel with the small inductance, constitutes the ultra-short wave oscillatory circuit. A small induction coil has been used for running the discharge-tube.

3. *Experimental Method and Results.*

In Table I. are given some of the experimental results giving the values of the dielectric constants of the ionized gases in the different discharge-tubes.

The dielectric constant can be easily calculated from the measurements of the wave-lengths when the discharge current is off and on. If L be the inductance of the loop and C the capacity of the condenser in the oscillatory circuit, the wave-length is given by

$$\lambda = A\sqrt{L.C.}\ *,$$

where A is a constant.

The value of C is changed to C′ when the medium between the plates of the condenser is ionized, so that the

TABLE I.

Ionized medium.	Tube current (m.a.)	λ (cm.).	$\Delta\lambda$ (cm.).	$K=\frac{\lambda'^2}{\lambda^2}$.	N.
Chlorine	4·5	438·0	2·4	·988	$1·76\times10^6$
Bromine	·5	431·2	6·6	·969	$4·71\times10^6$
Iodine	1·5	435·0	3·0	·987	$1·94\times10^6$
Thalium	4·0	427·4	3·0	·986	$2·16\times10^6$

N.B.—In calculating the dielectric constant of the ionized gases inside the discharge tubes, the correction for the glass sides of the tubes has not been made.

wave-length is modified. The modified wave-length is then given by

$$\lambda' = A\sqrt{L.C'}.$$

Thus the dielectric constant of the ionized medium can be determined from

$$K=\frac{C'}{C}=\frac{\lambda'^2}{\lambda^2}.$$

Since, according to Eccles-Larmor theory,

$$K=1-\frac{4\pi Ne^2}{m(\omega^2+\nu^2)},$$

* We have neglected the interaction of the valve on the oscillatory circuit. The interaction is negligible, since the value of C is much larger than the grid-plate capacity of the valve.

we get $$\frac{\lambda^2-\lambda'^2}{\lambda^2}=\frac{4\pi Ne^2}{mc^2}\cdot\lambda^2,$$

neglecting the collisional frequency ν, or

$$\frac{\lambda^2-\lambda'^2}{\lambda^4}=\frac{4\pi Ne^2}{mc^2}.$$

The electron-density N can thus be calculated if we can neglect ν. The calculated values neglecting ν are also incorporated in Table I.

4. *Dependence of the Dielectric Constant on the Ionic Concentration.*

The experiments have been carried out with discharge-tubes (chlorine, iodine, and thallium). The modified

TABLE II.

Tube current (m.a.)	$\Delta\lambda$ (cm.).	$\frac{\lambda'^2}{\lambda^2}=$K.	$\frac{\lambda^2-\lambda'^2}{\lambda^4}=\frac{4\pi Ne^2}{mc^2}$.
(*a*) Chlorine tube :			
4·0	2·4	·9891	$23{\cdot}88\times10^{-4}$
3·0	2·0	·9908	$19{\cdot}66\times10^{-4}$
2·0	1·2	·9945	$12{\cdot}17\times10^{-4}$
1·0	·6	·9972	$6{\cdot}085\times10^{-4}$
(*b*) Thallium tube :			
3·0	2·6	·9886	$26{\cdot}61\times10^{-4}$
2·0	1·8	·9917	$18{\cdot}09\times10^{-4}$
1·5	1·4	·9936	$13{\cdot}78\times10^{-4}$
1·0	1·0	·9954	$8{\cdot}56\times10^{-4}$
·5	·6	·9972	$5{\cdot}70\times10^{-4}$
(*c*) Iodine tube :			
1·2	2·0	·9908	$20{\cdot}05\times10^{-4}$
·9	1·8	·9917	$17{\cdot}72\times10^{-4}$
·5	1·8	·9917	$17{\cdot}72\times10^{-4}$

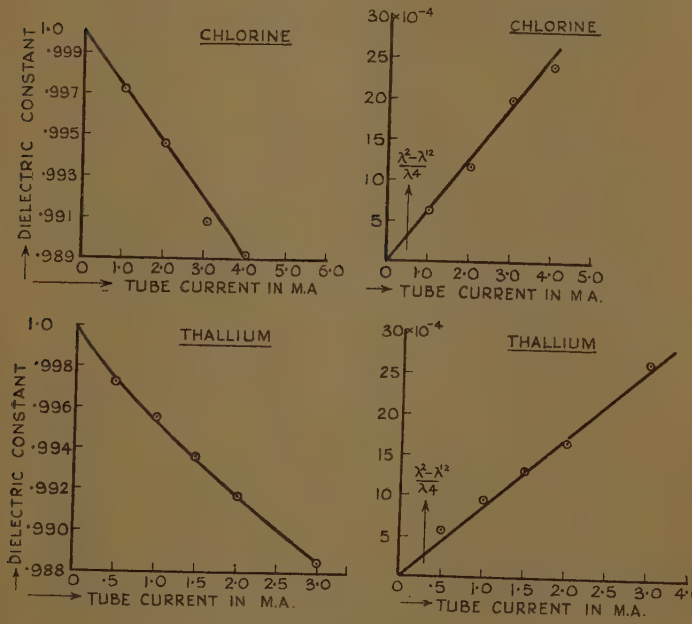
wave-length has been measured for various values of the tube current. The tube current has been varied by inserting suitable discharge-tubes as resistances in the circuit. The dielectric constant of the ionized gas in the chlorine and thallium tubes has been found to diminish below unity, approximately proportionately with the increase of the tube current. In the case of the iodine tube, however, the dielectric constant has been found to be more or less independent of the tube current.

The experimental results are given in Table II. and are graphically shown in fig. 3.

5. *Variation of the Dielectric Constant of an Ionized Gas with the Wave-length of the Oscillation.*

The dielectric constant of the ionized medium has been measured for a certain range of wave-lengths. The

Fig. 3.



range of wave-length employed lies between 3·98 m. to 5·4 m. The reduction in the value of the dielectric constant below unity has been found to increase and then to gradually decrease. The experimental results are given in Table III. and illustrated in fig. 4.

It is to be noted that the values of the dielectric constant have always been less than unity. From fig. 4 it is evident that the reduction of the dielectric constant is nearly proportional to the square of the wave-length

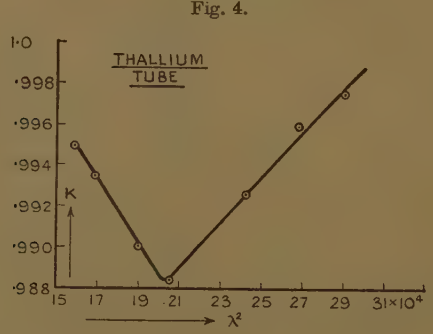
up to a certain wave-length, where there is an unmistakeable turning-point in the dielectric constant curve. The dielectric constant then begins to increase, thus approaching the value of unity

TABLE III.

Thallium tube.

λ (cm.).	λ^2.	$\Delta\lambda$ (cm.).	$K=\frac{\lambda'^2}{\lambda^2}$.
398·2	1586×10^2	1·0	·9949
411·2	1690 „	1·4	·9936
436·0	1901 „	2·2	·9900
453·4	2056 „	2·6	·9886
492·2	2423 „	1·8	·9926
518·4	2686 „	1·2	·9959
538·0	2895 „	·6	·9977

Fig. 4.



This anomalous increase of the dielectric constant has also been observed by Mitra and Banerjee, who have attributed it to the conductivity acquired by the ionized gas. Such an anomaly has also been observed in the experiments with an electron atmosphere in a screen-grid tetrode. (The results of our experiments with a tetrode valve are published elsewhere.) Owing to the bombardment of the electrons against the anode, the medium inside a screen-grid valve may acquire finite

conductivity. Usually, however, in a high vacuum valve, the collisional frequency is neglected and the conductivity regarded as zero. This interesting turning-point in the curve showing the relation between the dielectric constant and the wave-length has been repeatedly observed and is not believed to be spurious.

The curves, on the other hand, strongly suggest some kind of resonance effect. It may be connected with the free period of plasmoidal oscillations discussed by Tonks and Langmuir. No definite conclusion can, however, be reached as to the interpretation of these experimental results before more investigations are carried out along this line.

6. *Summary.*

Measurements of dielectric constants of some ionized gases (chlorine, bromine, iodine, and thallium vapour) have been made, avoiding the formation of ionic sheaths. The dielectric constant has always been found to be less than unity. In the case of the chlorine and thallium tubes the reduction in the value of the dielectric constant has been found to increase approximately proportionately with the increase of the discharge current, thus furnishing experimental evidence of the Eccles-Larmor theory. In the case of the iodine tube, however, the change in the dielectric constant has been found to be practically independent of the discharge current.

When the wave-length has been changed, keeping the tube current constant, the dielectric constant of the ionized gas has been found to decrease almost proportionately with the square of the wave-length up to a certain wave-length, thus showing agreement with the Eccles-Larmor formula up to that point. Beyond this wave-length the dielectric constant has been found to increase. Some kind of resonance effect has been suggested to explain this experimental result.

Physics Laboratory,
Dacca University.
October, 1936.

LXXXVI. *On the Representation of Non-Linear Motions by Series of Damped Time Exponentials.* *By* R. A. Frazer, *B.A., B.Sc., F.R.Ae.S., of the Aerodynamics Department, National Physical Laboratory* *.

1. *Subject of the Note.*

It is known that with linear dynamical systems which are not conservative, any free motion about the equilibrium configuration is stable provided the real parts of the characteristic exponents (roots of the determinantal equation) are all negative. When this condition is satisfied, and when the characteristic exponents are all distinct, the most general motion about equilibrium can be expressed mathematically by suitable linear combinations of time exponentials, each representing a damped constituent motion.

It does not appear to be generally recognized that with non-linear dynamical equations the possibility of a development of the solution in the form of *series* of sums of damped exponentials does not necessarily imply stability. In the present note some simple examples are given of non-linear motions which increase indefinitely, although they can be represented by series containing solely subsident time exponentials.

2. *Preliminary Remarks on the Method.*

Series of the particular type referred to are encountered when Poincaré's method [1] of expansion of the solution in powers of parameters is used in relation to dynamical systems which are definitely stable when all the non-linearities are supposed removed. Some preliminary remarks follow regarding the general sense in which the method will be applied.

Consider firstly a system of linear Lagrangian equations of motion with constant coefficients. These can be expressed concisely as

$$U_i = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where U_i is homogeneous and linear in the m generalized coordinates q and in the generalized velocities $\dot{q}$ and

* Communicated by E. F. Relf, F.R.S. The writer is indebted to the Aeronautical Research Committee for permission to publish the paper.

accelerations $\ddot{q}$, while i is to be regarded as a typical suffix ranging from 1 to m. For simplicity it will be assumed that the determinantal equation $\Delta(\lambda)=0$ has $n=2m$ distinct roots $\lambda_1, \lambda_2, \ldots, \lambda_n$. The general solution of the system of differential equations (1) can then be written

$$q_i = \sum_{j=1}^{n} c_j k_{ij} e^{\lambda_j t},$$

where c_j are arbitrary constants and k_{ij} are modal coefficients.

Equations (1) are reducible to a simple canonical form as follows. Introduce n new dependent variables $\alpha_1, \alpha_2, \ldots, \alpha_n$, defined in terms of the quantities q and $\dot{q}$ by the n relations

$$\left.\begin{aligned} q_i &= \sum_{j=1}^{n} k_{ij}\alpha_j, \\ \dot{q}_i &= \sum_{j=1}^{n} \lambda_j k_{ij}\alpha_j. \end{aligned}\right\} \quad \ldots\ldots \quad (2)$$

Then the m equations (1) are equivalent to the n first-order equations

$$\dot{\alpha}_j - \lambda_j \alpha_j = 0. \quad \ldots\ldots \quad (3)$$

Next consider a system of non-linear differential equations

$$\mathrm{U}_i = \mathrm{RF}_i, \quad \ldots\ldots \quad (4)$$

in which U_i is as previously defined and F is assumed to be rational, integral, and at least of the second degree in the quantities q and $\dot{q}$, but independent of the time variable and of the accelerations $\ddot{q}$. The coefficient R, which is for simplicity supposed to be the same for all the equations *, will be referred to as the *non-linearity parameter*. When the dependent variables are changed from q to α, in accordance with (2), equations (4) evidently reduce to the canonical form

$$\dot{\alpha}_j - \lambda_j \alpha_j = \mathrm{RG}_j, \quad \ldots\ldots \quad (5)$$

where G_j is rational and integral in the variables α. In relation to this form of the dynamical equations an "initial disturbance" is specified by assigned initial values, say A, for the variables α.

* In a more general treatment the coefficients of the separate terms in each F_i would be treated as distinct non-linearity parameters.

The particular method of solution to be illustrated is based on the consideration that a form of the solution of equations (5), which is certainly valid for sufficiently small values of R and t, is given by

$$\alpha_j(t)=A_j e^{\lambda_j t}+R\alpha_{1j}(t)+R^2\alpha_{2j}(t)+R^3\alpha_{3j}(t)+\ldots \quad . \quad (6)$$

The leading terms of the n series (6) represent the solution of the linear equations (3) appropriate to the conditions $\alpha_j=A_j$ at $t=0$. Also, if the initial disturbance is assumed to be independent of R, the functions α_{sj} must all be zero at $t=0$.

To obtain the recurrence relations which formally determine the functions α_{sj}, the trial solution (6) is substituted in (5), and the total coefficients of the successive powers of R are equated to zero. The first set of equations so obtained is

$$\alpha_{1j}-\lambda_j\alpha_{1j}=G_j(A_1e^{\lambda_1 t},\ A_2e^{\lambda_2 t},\ \ldots,\ A_ne^{\lambda_n t}).$$

Hence, since $\alpha_{1j}=0$ at $t=0$,

$$\alpha_{1j}=e^{\lambda_j t}\int_0^t e^{-\lambda_j t}G_j(A_1e^{\lambda_1 t},\ A_2e^{\lambda_2 t},\ \ldots,\ A_ne^{\lambda_n t})\,dt. \quad . \quad (7)$$

The coefficients of the higher powers of R in (6) are determinable in succession on a similar basis.

It is not intended in the present paper to enter fully into the analytical details of this method of solution. However, there are certain general points connected with the mathematical structure of the solution which may be noted. Suppose $\alpha_1^{r_1}\ \alpha_2^{r_2}\ \ldots\ \alpha_n^{r_n}$ to be a typical product in $G_j(\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_n)$. Then by (7) this product will normally contribute to α_{1j} the term

$$A_1^{r_1}A_2^{r_2}\ldots A_n^{r_n}(e^{\theta t}-e^{\lambda_j t})/(\theta-\lambda_j),$$

where

$$\theta\equiv r_1\lambda_1+r_2\lambda_2+\ldots+r_n\lambda_n.$$

However, if it so happens that $\theta=\lambda_j$, then the term contributed is

$$A_1^{r_1}A_2^{r_2}\ldots A_n^{r_n}te^{\lambda_j t}.$$

This is sufficient to show that the terms of the series (6) will in general be linear combinations of time exponentials whose indices are sums of multiples of the n characteristic exponents $\lambda_1,\ \lambda_2,\ \ldots,\ \lambda_n$, but that in exceptional cases the terms may also involve polynomials of t.

When the real parts of the characteristic exponents are all negative (linear motion stable) the time exponentials which occur in the series (6) are necessarily all of the damped type. The series appropriate to any given origin of t will in general be convergent only in a certain limited interval of t, which itself depends on the magnitude of R and on the initial conditions. However, by successive changes of the time origin, and use of the corresponding series appropriate to the several origins, a convergent representation of the complete motion is possible over any assigned range of t. In the particular case of a periodic motion this analytic continuation may be unnecessary, since a single series of damped exponentials may well provide a valid representation over at least one complete period. If required, a representation of the motion valid for all time can then be derived by the methods of Fourier analysis.

The examples which follow are intended merely to illustrate the numerical application of the method, and they relate to very simple non-linear differential equations, which are also capable of exact solution. The appropriate solutions obtained by expansions in powers of R will be compared with the exact solutions.

3. *Example No.* 1.—*Ordinary Differential Equation of the First Order.*

Let the solution of

$$\dot{q}+\lambda q=\mathrm{R}q^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

be required appropriate to the initial condition $q=\mathrm{Q}$ at $t=0$. The constants R and λ are assumed to be positive.

When $\mathrm{R}=0$ the solution is simply $q=\mathrm{Q}e^{-\lambda t}$. Hence if (8) is viewed as a "dynamical equation" the linear motion is subsident for all "disturbances" Q. In the present case the equation is also directly integrable for a general R by separation of the variables. Thus

$$q(t)=\frac{\mathrm{Q}e^{-\lambda t}}{1-\dfrac{\mathrm{RQ}}{\lambda}(1-e^{-\lambda t})} \quad . \quad . \quad . \quad . \quad (9)$$

If $\mathrm{RQ}<\lambda$ the function $q(t)$ decreases to zero as t increases from 0 to ∞ : if $\mathrm{RQ}=\lambda$ the function is constant and equal

to Q for all t: lastly, if $RQ>\lambda$, $q(t)$ changes from $+\infty$ to $-\infty$ on passage of t through the value t_0 given by $e^{\lambda t_0}=1-\frac{\lambda}{RQ}$. The graphs of $q(t)$ when R=0·5, 1·0, and 1·1, and Q=λ=1, are shown by the dotted curves in fig. 1: the asymptote for R=1·1 lies at $t_0=2{\cdot}398$.

In the present case a series solution $S(t)$ in ascending powers of R can be written down immediately from (9). Thus

$$S(t)=Qe^{-\lambda t}+\frac{RQ^2}{\lambda}e^{-\lambda t}(1-e^{-\lambda t})+\frac{R^2Q^3}{\lambda^2}e^{-\lambda t}(1-e^{-\lambda t})^2+\ldots \quad . \quad . \quad . \quad (10\,a)$$

or, say,

$$S(t)=a_0+Ra_1+R^2a_2+R^3a_3+\ldots \quad . \quad (10\,b)$$

This series is convergent for all positive t when $RQ<\lambda$, and for $0\leqslant t\leqslant t_0$ when $RQ>\lambda$. Hence in all cases it provides a convergent representation of $q(t)$ up to the nearest singularity. The full curves in fig. 1 show the successive approximations to $q(t)$ given by $S(t)$ when R=1·1 and Q=λ=1. The numbers against the curves indicate the number of terms of $S(t)$ included.

It is, of course, possible to construct (10 *a*) by the method of § 2 without a previous knowledge of the exact solution. Thus, if (10 *b*) is substituted in (8), the total coefficients of the separate powers of R will be zero if

$$\dot{a}_0=-\lambda a_0,$$
$$\dot{a}_1=-\lambda a_1+a_0^2,$$
$$\dot{a}_2=-\lambda a_2+2a_0a_2,$$

and so on. These relations, taken in conjunction with the conditions a_0=Q and $a_i=0 (i\neq 0)$ at $t=0$, lead to the series (10 *a*).

Some further results for the case R=1·1 and Q=λ=1 are given in Table I., which illustrate the use of successive series in the computation of a solution. The series (10 *a*) was first used relative to the time origin $t=0$ to obtain the value of $q(t)$ at $t=0{\cdot}5$. This terminal value, $q(0{\cdot}5)$, was then substituted for Q in (10 *a*) and the series so obtained was used, relative to the new origin of time, $t=0{\cdot}5$, for computation up to $r=1$. The process was continued

Fig. 1.

EXAMPLE. Nº 1 (§ 3).

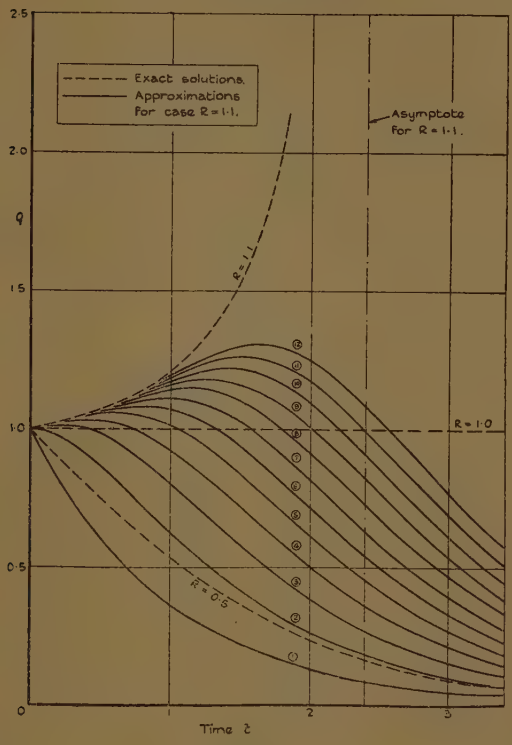


TABLE I.

Values of $q(t)$ computed by Continuation.

(Case $Q=\lambda=1$ and $R=1{\cdot}1$.)

t.	Value of $q(t)$ calculated by (9).	Approximate value calculated by series.	Number of terms used in each series.
0·5	1·0695	1·0691	11
1·0	1·2074	1·2071	13
1·5	1·5344	1·5331	15
2·0	2·7725	2·7636	22

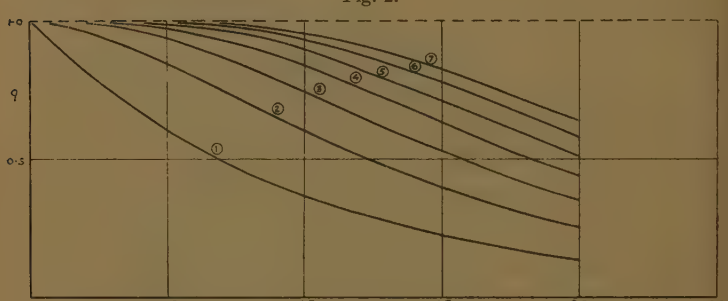
up to $t=2{\cdot}0$, a new origin of t being adopted every 0·5 sec.

When $RQ=\lambda$ the function $q(t)$ reduces to a constant Q, and (10 *a*) then yields the curious identity

$$1=e^{-\lambda t}+e^{-\lambda t}(1-e^{-\lambda t})+e^{-\lambda t}(1-e^{-\lambda t})^2+\ldots \qquad (11)$$

This series converges for all finite positive t, but the convergence is naturally extremely slow when t is large. The successive approximations to $q(t)=1$ derived from the series are shown in fig. 2. It may be noted that the possible positions of "equilibrium" $q=Q$ defined by the "dynamical" equation (8) are given by the condition $0=-\lambda Q+RQ^2$. Hence either $Q=0$ or $RQ=\lambda$. The

Fig. 2.



Representation of constant by series of subsident time exponentials (see Equation (11)).

present method thus leads to a representation not only of motion, but also of equilibrium, by series composed of damped time exponentials.

4. *Example No. 2.—Ordinary Differential Equation of the Second Order.*

A somewhat more complicated example is provided by the equation

$$\ddot{q}+3\dot{q}+2q=2R(q\dot{q}+q^2), \quad \ldots \quad (12)$$

in which R is assumed to be positive. A first integral of (12) is

$$\dot{q}+q-Rq^2+Ae^{-2t}=0, \quad \ldots \quad (13)$$

and if $z=qe^t$ and $\tau=1-e^{-t}$, (13) can be written

$$\frac{dz}{d\tau}=Rz^2-A,$$

or, say,

$$\frac{dz}{d\tau}=R(z^2-\theta^2). \quad . \quad . \quad . \quad . \quad . \quad (14)$$

On integration this last equation yields

$$z/\theta=\coth R\theta(\tau_0-\tau),$$

in which τ_0 is arbitrary. The general solution of (12) may accordingly be taken as

$$q=\theta e^{-t}\coth\{R\theta(e^{-t}-e^{-t_0})\}, \quad . \quad . \quad . \quad (15)$$

where θ and t_0 are arbitrary constants. If $q=Q$ and $\dot{q}=Q_1$ at $t=0$, then θ and t_0 are given by

$$R\theta^2=RQ^2-Q-Q_1, \quad . \quad . \quad . \quad . \quad . \quad (16\,a)$$

$$R\theta e^{-t_0}=R\theta-\frac{1}{2}\log\left(\frac{Q+\theta}{Q-\theta}\right). \quad . \quad . \quad (16\,b)$$

For simplicity it will be assumed that the initial disturbance is such that θ and t_0 are both real *. These conditions are satisfied if, for instance, $Q=0{\cdot}2$, $Q_1=0$, and $R=10{\cdot}0$: in this case

$$\theta=\sqrt{2}/10=0{\cdot}141421,$$

$$t_0=0{\cdot}976,$$

and the solution is

$$q=0{\cdot}141421e^{-t}\coth\{1{\cdot}41421(e^{-t}-e^{-0{\cdot}976})\}. \quad . \quad (17)$$

The graph, which has an asymptote at $t=t_0$, is the curve shown dotted in fig. 3.

To obtain a series solution in ascending powers of R, the simplest procedure in the present case is to make use of the known first integral and to derive in the first place a direct development of $z\equiv qe^t$ as a power series in $\tau\equiv 1-e^{-t}$. Thus by Maclaurin's theorem

$$q\equiv z(\tau)e^{-t}=\left[(z)_{\tau=0}+\tau\left(\frac{dz}{d\tau}\right)_{\tau=0}+\frac{\tau^2}{2}\left(\frac{d^2z}{d\tau^2}\right)_{\tau=0}+\ldots\right]e^{-t}.$$

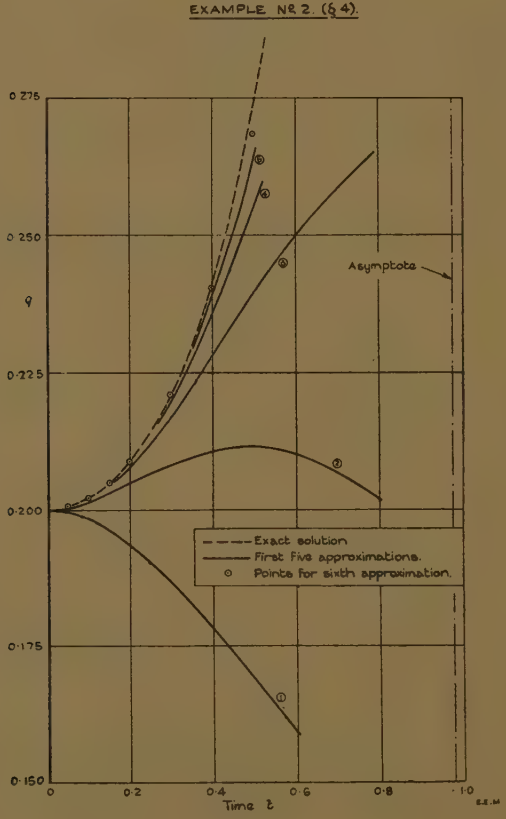
$$. \quad . \quad . \quad (18)$$

* The formulæ require some modification in other cases.

But by (14) and (16 *a*)

$$\frac{dz}{d\tau} = R(z^2 - Q^2) + Q + Q_1 = f(z), \quad \text{say.}$$

Fig. 3.



Hence

$$\frac{d^2z}{d\tau^2} = f\frac{df}{dz}\ ; \quad \frac{d^3z}{d\tau^3} = f^2\frac{d^2f}{dz^2} + f\left(\frac{df}{dz}\right)^2\ ; \quad \text{and so on.}$$

Moreover,

$$(f)_{\tau=0}=Q+Q_1\,;\quad \left(\frac{df}{dz}\right)^{\cdot}_{\tau=0}=2RQ\,;$$

$$\left(\frac{d^2f}{dz^2}\right)_{\tau=0}=2R\,;\quad \left(\frac{d^3f}{dz^2}\right)_{\tau=0}=0,\ \text{etc.}$$

In this way the coefficients in (18) are determined successively. The series obtained gives on rearrangement in ascending powers of R

$$q=[a_0(\tau)+Ra_1(\tau)+R^2a_2(\tau)+\ldots]e^{-t}, \qquad (19)$$

in which the first six coefficients are

$$a_0=Q+\tau(Q+Q_1),$$

$$a_1=\tau^2Q(Q+Q_1)+\frac{1}{3}\tau^3(Q+Q_1)^2,$$

$$a_2=\frac{2}{3}\tau^3Q^2(Q+Q_1)+\frac{2}{3}\tau^4Q(Q+Q_1)^2+\frac{2}{15}\tau^5(Q+Q_1)^3,$$

$$a_3=\frac{1}{3}\tau^4Q^3(Q+Q_1)+\frac{11}{15}\tau^5Q^2(Q+Q_1)^2+\frac{34}{90}\tau^6Q(Q+Q_1)^3+\frac{34}{630}\tau^7(Q+Q_1)^4,$$

$$a_4=\frac{2}{15}\tau^5Q^4(Q+Q_1)+\frac{52}{90}\tau^6Q^3(Q+Q_1)^2+\frac{360}{630}\tau^7Q^2(Q+Q_1)^3+\frac{124}{630}\tau^8Q(Q+Q_1)^4+\frac{124}{5670}\tau^9(Q+Q_1)^5,$$

$$a_5=\frac{4}{90}\tau^6Q^5(Q+Q_1)+\frac{228}{630}\tau^7Q^4(Q+Q_1)^2+\frac{384}{630}\tau^8Q^3(Q+Q_1)^3+\frac{2144}{5670}\tau^9Q^2(Q+Q_1)^4+\frac{5528}{56700}\tau^{10}Q(Q+Q_1)^5+\frac{5528}{623700}\tau^{11}(Q+Q_1)^6.$$

The approximations to (17) obtained in the case $Q=0{\cdot}2$, $Q_1=0$, $R=10$ are shown by the full curves in fig. 3. All the terms of the series (19) are composed of sums of subsident time exponentials.

Once the first five or six terms of (19) have been obtained for general values of Q and Q_1 it is possible to calculate $q(t)$ over a wide range of t by continuation. As an illustration a rough value for $q(t)$ at $t=0{\cdot}6$ will be obtained for the case just considered by continuation in steps of $t=0{\cdot}2$. For this purpose it is necessary to compute both $q(t)$ and $\dot{q}(t)$ at the end of each time interval in order to provide the initial values Q and Q_1 for the next time interval. Thus (19) must be used in conjunction with the derived series

$$\frac{dq}{dt}=-e^{-t}[a_0(\tau)+\mathrm{R}a_1(\tau)+\mathrm{R}^2a_2(\tau)+\ldots]$$

$$+e^{-t}\frac{d}{dt}[a_0(\tau)+\mathrm{R}a_1(\tau)+\ldots],$$

or, since

$$\frac{d\tau}{dt}=e^{-t},$$

$$\dot{q}=-q+e^{-2t}\left(\frac{da_0}{d\tau}+\mathrm{R}\frac{da_1}{d\tau}+\mathrm{R}^2\frac{da_2}{d\tau}+\ldots\right).$$

The approximate values for $q(t)$ obtained in this way are compared below (Table II.) with the values given by the exact solution (17).

TABLE II.

t.	Value of $q(t)$	
	By continuation.	By exact solution.
0·2	0·20876	0·20877
0·4	0·24125	0·24133
0·6	0·32474	0·32529

5. *Growing Oscillations Represented by Series of Damped Oscillations.*

The representation of an increasing oscillatory function by a series composed of decreasing oscillatory functions is illustrated by

$$e^{\mu t}\cos\omega t=e^{-(\lambda-\mu)t}\cos\omega t+e^{-(\lambda-\mu)t}(1-e^{-\lambda t})\cos\omega t$$

$$+e^{-(\lambda-\mu)t}(1-e^{-\lambda t})^2\cos\omega t+\ldots,$$

in which it is assumed that $\lambda-\mu \geqslant 0$. This identity is an obvious generalization of (11).

It is also possible to express oscillatory functions by series of pure subsident exponentials. For instance, the differential equation

$$\ddot{q}+\left(\alpha+\frac{1}{2}\right)\dot{q}\{1-\mathrm{R}(q^2+\dot{q}^2)\}+q=0, \quad . \quad . \quad (20)$$

is satisfied by $q=\sin t$ when $\mathrm{R}=1$, and when $\mathrm{R}=0$ the characteristic exponents are $-\alpha$ and $-1/\alpha$. Hence, if the method of § 2 were applied, a representation of $\sin t$ would be obtained as a series composed of terms of the type

$$\exp\left\{-\left(m\alpha+\frac{n}{\alpha}\right)t\right\}.$$

The treatment obviously admits of considerable generalization, and it may be presumed that oscillatory motions of a very general type are expressible by series of damped exponentials.

6. *Suggested Applications to Problems of Applied Mathematics.*

Some concluding remarks may be useful regarding certain problems of applied mathematics in which series of the type considered may be expected to arise. In recent years non-linear oscillation theory has been much developed in relation to radio-technique [2], but it also has an important bearing on aerodynamical questions, such as wing-flutter and the stability of viscous fluid motion. It is with these aerodynamical applications that the writer is mainly concerned.

In the usual approximate treatment of wing-flutter * an aeroplane wing is treated as a linear dynamical system with a finite number of degrees of freedom or modes of distortion, and the commonly accepted criterion for stability of the wing is the absence of characteristic exponents with positive real parts. However, it is a recognized fact that when large air disturbances are encountered a wing may be set into sustained oscillation even though it is stable on the basis of the foregoing criterion. The mathematical problem here is of the same

* See, for example, Ref. [3].

general nature as that represented by, for instance, equation (20), although it is, of course, vastly more complicated. One further aspect of the subject of flutter may be noted. Linear oscillation theory requires that when flutter occurs the amplitude of the wing-oscillations shall increase indefinitely. However, in practice the ultimate outcome of flutter is often a sustained oscillation ; this limitation of the amplitude is essentially a phenomenon of non-linear mechanics.

As regards a possible connexion with the subject of fluid motion, any suggestions that can be made at present are necessarily of a purely speculative character. It is known that the method of expansion in powers of a parameter can sometimes be applied to partial differential equations [4], [5]. To provide a simple example, the exactly integrable equation

$$\nu \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x}$$

can be treated successfully in this way, and numerical solutions obtained by expansion in powers of $1/\nu$ will be found to accord satisfactorily with the exact solutions. The equation is of academic interest only, but it will be recognized as containing some of the terms of the equations of flow of a viscous fluid. If, in fact, a similar method is applicable to the complete hydrodynamical equations, it will lead to a representation of the flow by series in which the time variable appears in the form of time exponentials. The results of the present paper suggest that such series may well represent turbulent motions, even when all the characteristic exponents (in this case infinite in number) have negative real parts.

Some support for the foregoing conclusion is indirectly furnished by the investigation of G. I. Taylor [6] on the stability of a viscous liquid enclosed between two steadily rotating cylinders. The criterion adopted by Taylor for the onset of instability is the presence of a zero characteristic exponent, corresponding effectively to a neutral condition of flow in which " vortex banding " appears. However, if the views now put forward are correct, it is probable that with sufficiently large disturbances turbulence could be induced even although the real parts of the characteristic exponents were negative.

Under such conditions a turbulent régime, once established, would persist below the critical points predicted by Taylor's first-order theory. This was, in fact, confirmed by the experiments carried out by J. W. Lewis [7] subsequent to Taylor's original investigation. A phenomenon similar to that discussed by Taylor can be observed in the case of rotary decaying flow of a liquid within a cylindrical container of circular section. If the complete system is initially in moderately fast steady rotation, and if the rotation of the *container* is abruptly stopped, the ensuing decaying motion of the fluid exhibits definite vortex banding resembling that described by Taylor. On the other hand, if the initial rate of rotation of the complete system is great, the decaying flow after arrest of the container will temporarily exhibit turbulent characteristics. The mathematical analysis of this very complex problem leads to a representation of the flow by series of time exponentials, all of which are definitely of the damped type. Hence it may be concluded that temporary turbulence at least can be represented by series of the special type considered.

In conclusion, the writer wishes to express his indebtedness to Miss S. W. Skan, of the Aerodynamics Department, National Physical Laboratory, who carried out all the numerical computations given in the present paper.

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LXXXVII. *Deflexion of a Current in a Gas by a Magnetic Force.* *By* J. S. Townsend, *M.A., F.R.S., Wykeham Professor of Physics, Oxford* *.

THE method of finding the mean free path of an electron in a gas and the mean loss of energy of electrons in collisions with molecules of the gas, in accordance with the general theory of the motion of charged particles in a gas under the action of a uniform electric force, depends on the results of experiments from which the mean energy of the electrons $mV^2/2$ and the mean velocity W in the direction of the force are deduced.

Without assuming any particular values of the constants that occur in the general equations of motion, it may be seen that a steady state of motion is obtained where the mean energy and the mean velocity depend only on the ratio of the electric force Z to the pressure p of the gas.

In order to compare the steady motions of electrons in different gases, the numerical constants that occur in the equations of motion may be estimated by approximate calculations, and the values of V and W thus obtained may be used to find the mean free paths and the mean loss of energy of electrons in collisions to the same degree of accuracy.

It is of interest, however, to deduce the values of V and W from the results of the experiments, by equations of motion where the mean velocities of the electrons are calculated in accordance with the theory of motion of charged particles of very small mass moving with large velocities of agitation, where it may be assumed that all directions of motion of the particle after a collision with a molecule of the gas are equally probable, and the lengths of the free paths of the particles are independent of the velocity of agitation. It may also be assumed that the particles lose small amounts of energy in the collisions which are restored by the displacement in the direction of the force.

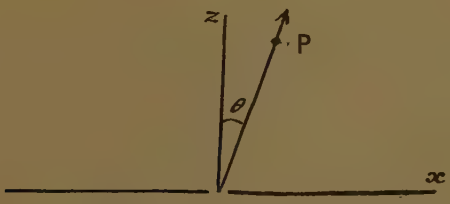
The mean energy of the electrons $mV^2/2$ is deduced from the results of experiments on the lateral divergence of a narrow stream of electrons in moving through a uniform electric field, and it has been shown that under

* Communicated by the Author.

the above conditions with regard to free paths of very small particles the energy $mV_1^2/2$ obtained directly from the experiments is $(3a/2b)\times mV^2/2$, where a and b are numerical factors depending on the distribution of the velocities of agitation of the electrons *.

For the Maxwellian distribution the factor a is ·92 and b is $3/2\times\cdot92$, so that the mean energy of agitation is $mV_1^2/2$, but in general the velocities are not so widely distributed, and the mean energy is in many cases about $\cdot85\times mV_1^2/2$.

The mean velocity W has been deduced from observations of the deflexion of a stream of electrons by the action of a magnetic force. In the following investigation of the deflexion of a stream of electrons by a small magnetic force H, the electric force Z is taken to be in the direction of the axis z, the magnetic force in the direction of the axis y, and the deflexion in the direction of the



axis x. Thus when a group of electrons passes through a narrow slit in a plate in the plane (xy), the centre P of the group moves in a line inclined at a small angle θ to the axis of z as shown in the figure.

The value of θ as found by the equations † of motion which have usually been adopted is given by the equation $\tan\theta=\mathrm{H}.\mathrm{W}/\mathrm{Z}$, W being the mean velocity of the electrons in the direction z. This equation appears to be in accordance with elementary dynamics, as it shows that the centre of the group moves along the diagonal of the parallelogram of the forces $He\mathrm{W}$ in the direction x, and Ze in the direction z. This agreement, however, is not conclusive evidence of the accuracy of the equation, as it cannot be assumed that the ratio of the mean

* Phil. Mag. vol. xxiii. p. 48 (March 1937).
† ' Electricity in Gases,' Chap. III.

velocities of a group of electrons in the directions of the axes x and z is equal to the ratio of the forces, since HeW is an average value of different forces acting on an electron, and Ze is a constant force acting continuously.

It has already been shown that the equations for the mean velocities of charged particles in a uniform electric field which have been usually adopted depend on a simplification which introduces inaccuracies when the equations are applied to the motion of electrons, so that it is desirable to determine the deflexion θ due to a small magnetic force taking into consideration the above conditions with regard to free paths of small particles in a gas, and the effect of collisions on the direction of motion of the particles.

An expression for the deflexion θ in accordance with these conditions may be found by a method similar to that * which I have already given to determine the velocity W in terms of the mean free path and the mean velocity of agitation.

The mean velocity u of electrons in the direction of the axis of x is found from the expression for the mean velocity $\bar{u}_1$ of a group which has the same velocity of agitation U_1.

In order to find the mean velocity $\bar{u}_1$, it may be supposed each of the electrons in the group traverses a large number C of free paths of lengths l_1, l_2, . . . l_c, in the time $l C/U_1$, which is approximately the same for each electron, l being the mean free path: also, it may be supposed that the free paths of length l_1 begin at the same time at a point O, where all directions of motion of the electrons are equally probable.

Let the origin of coordinates be at O and x, y, and z the ordinates of an electron at any time t while traversing the path of length l_1. The accelerations of the electron in the directions of the axes are given by the equations

$$d^2x/dt^2 = g\,dz/dt,\quad d^2y/dt^2 = 0,$$

and

$$d^2z/dt^2 = f - g\,dx/dt,$$

f being the acceleration Ze/m and g the angular velocity He/m. In order to find the principal terms in the integrals of these equations the following conditions are taken into consideration. The forces Ze and HeU_1 are small,

* Phil. Mag. vol. xxii. p. 145 (July 1936).

and it may be assumed that the velocities fl_1/U_1 and gl_1, acquired in the time l_1/U_1, are small compared with U_1. Also, when either of the forces Z or H is zero, the mean displacement of the group in the direction of the axis of x is zero, so that the product Z.H must occur in each term of the expression for the mean velocity $\bar{u}_1$, and since Z and H are small it is only necessary to consider the principal term which is proportional to $f \times g$.

Neglecting terms in g^2, the velocities of the electron at any time t are

$$\left.\begin{aligned} &dx/dt = u_0 + gw_0t + f \,.\, gt^2/2, \; dy/dt = v_0, \\ \text{and} \quad &dx/dt = w_0 + (f - u_0g)t, \end{aligned}\right\} \quad . \quad . \quad (1)$$

u_0, v_0, and w_0 being the velocities of the electron at the origin, and the position of the electron is given by the ordinates

$$\left.\begin{aligned} &x = u_0t + gw_0t^2/2 + fgt^3/6, \; y = v_0t, \\ \text{and} \quad &z = w_0t + (f - gu_0)t^2/2. \end{aligned}\right\} \quad . \quad . \quad . \quad (2)$$

For a first approximation t may be taken to be l_1/U_1, at the end of each free path of length l_1, and with this simplification the mean velocities of the group of electrons are $f.g.l^2/U_1^2$ in the direction of the axis of x, and fl/U_1 in the direction of the axis of z, and the deflexion θ is given by the equation * $\tan\theta = H.W/Z$.

For a more accurate estimate of the mean velocities it is necessary to take into consideration the fact that all electrons starting in different directions from the origin O, with the same velocity of agitation U_1, do not complete the free paths of length l_1 in the same time l_1/U_1. The free paths are slightly curved, and if s be the length of the path traced out by an electron, measured from the origin, the velocity ds/dt of the electron at any time t is obtained by equating $(ds/dt)^2$ to the sum of the squares of the velocities in the directions of the axes as given by equation (1). The principal terms in the expression thus obtained for (ds/dt) are given by the equation

$$ds/dt = U_1(1 + w_0ft/U_1^2 - u_0fgt^2/2U_1^2), \quad . \quad . \quad (3)$$

and the time t_1 required to traverse a path of length l_1 is given by the integral of this equation, which is

$$l_1 = U_1t_1(1 + w_0ft_1/2U_1^2 - u_0fgt_1^2/6U_1^2). \quad . \quad . \quad (4)$$

* ' Electricity in Gases,' p. 102.

For a first approximation t_1 is l_1/U_1, and a second approximation is given by the equation

$$t_1=(1-w_0fl_1/2U_1^2+u_0fgl_1^2/6U_1^4)l_1/U_1. \quad . \quad . \quad (5)$$

The distances x, y, and z of the electron from the coordinate planes at the end of the free path of length l_1 are obtained by substituting this value of t_1 for t in equations (2).

The terms involving in the first powers of u_0 and w_0 do not contribute to the mean distances from the coordinate planes of the group of electrons which start in various directions from the origin, since the mean values of u_0 and of w_0 are zero. Also the mean values of u_0^2 and w_0^2 are $U_1^2/3$. The mean displacements x_1 and z_1 in the directions x and z of the group of electrons at the ends of the free paths of length l_1 are thus found to be

$$x_1=fgl_1^3/18U_1^3 \quad \text{and} \quad z_1=fl_1^2/3U_1^2, \quad . \quad . \quad (6)$$

and the mean displacements at the end of a large number c of free paths of different lengths are

$$\bar{x}_1=fgcl^3/3U_1^3 \quad \text{and} \quad \bar{z}_1=2fcl^2/3U_1^2, \quad . \quad . \quad (7)$$

since the mean of the cubes of a large number of free paths is $6l^3$ and the mean of the squares is $2l^2$, l being the mean free path *.

Since these are the mean displacements at the time lc/U_1, the mean velocities $\bar{u}_1$ and $\bar{w}_1$ of the group of electrons which have the same velocity of agitation U_1 are

$$\bar{u}_1=f.g.l^2/3U_1^2 \quad \text{and} \quad \bar{w}_1=2fl/3U_1. \quad . \quad . \quad (8)$$

Thus, if all the electrons had the same velocity of agitation, the mean magnetic deflexion θ would be given by the equations

$$\tan\theta=\bar{u}_1/\bar{w}_1=fl/2U_1=(3/4)\times HW/Z. \quad . \quad . \quad (9)$$

In order to find the mean deflexion when the velocities of agitation U are distributed about a mean value, it is necessary to substitute in equations (8) the mean values of $1/U^2$ and $1/U$ for $1/U_1^2$ and $1/U_1$.

For the Maxwellian distribution the mean value of $1/U^2$ is $3/V^2$, V^2 being the mean of the squares of the velocities of agitation, and the mean value of $1/U$ is $1{\cdot}38/V_0$. For this distribution of the velocities of

* ' Electricity in Gases,' Chap. III.

agitation the mean velocities $\bar{u}$ and $\bar{w}$ of the electrons are therefore

$$\bar{u}=fgl^2/V^2 \quad \text{and} \quad \bar{w}=\cdot 92\times fl/V_1, \quad . \quad . \quad . \quad (10)$$

and the magnetic deflexion is given by the equations

$$\tan\theta=gl/(\cdot 92\times V)=1\cdot 18\times H.W/Z. \quad . \quad . \quad (11)$$

In general the velocities of agitation of the electrons are distributed less widely than in the Maxwellian distribution, so that when θ is determined experimentally the upper and lower limits to the value of W are

$$4/3\times Z\tan\theta/H \text{ and } \cdot 85\times Z\tan\theta/H,$$

the exact value of the numerical constant depending on the distribution.

LXXXVIII. *Points of Congestion in the Periodontal Membrane. By* J. L. Synge, *Sc.D., Professor of Applied Mathematics, University of Toronto* †.

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1. *Introduction.*

In man and animals the tooth is attached to the socket by the thin periodontal membrane. In two previous papers ‡ I have discussed the equilibrium of the tooth under given applied forces, the argument being based on the assumption that the membrane may be regarded as an incompressible elastic solid. The thickness of the membrane is regarded as infinitesimal, and only principal parts of expressions are retained.

The assumption of incompressibility is violated when extra blood is drawn into the membrane from the circulatory system, or when it is squeezed out. We are familiar

† Communicated by the Author.

‡ Phil. Trans. Roy. Soc. A, ccxxxi. pp. 435–477 (1933); Phil. Mag. (7) xv. pp. 969–996 (1933). These papers will be referred to as T_1 and T_2 respectively.

with the way in which a tooth is displaced outwards when a state of inflammation exists: this must be attributed to an increase in volume in the membrane due to the increased circulation consequent on the inflammation.

The purpose of the present paper is to develop some of the consequences of inflammation or congestion in the membrane, idealizing the conditions so as to permit a mathematical treatment. Apart from the dental aspects of the problem, the work is of some general interest. because it gives a physical background to the problem of Dirichlet in a curved two-dimensional region.

2. *Equations for the Normal Membrane.*

Let us set down the fundamental equations †. valid for normal regions of the membrane, *i. e.*, uncongested regions where the membrane is incompressible. Greek suffixes having the range of values 1, 2, with summation over that range when repeated, let x^α be any system of curvilinear coordinates on the surface of the root of the tooth. Let the line-element of this surface be

$$ds^2 = b_{\alpha\beta}\,dx^\alpha\,dx^\beta. \quad . \quad . \quad . \quad . \quad . \quad (2.1)$$

Let $h(x^1, x^2)$ be the small thickness of the membrane and μ its rigidity, supposed constant.

The principal part of the stress in the membrane is a pressure p, a function of x^1, x^2; p satisfies the partial differential equation

$$\mathrm{B}^\alpha(h^3\,\mathrm{B}_\alpha p) = 12\mu\beta, \quad . \quad . \quad . \quad . \quad (2.2)$$

where B_α, B^α denote the operations of covariant and contravariant differentiation with respect to the metric (2.1), and β is the normal displacement of the point (x^1, x^2) of the root, counted positive when directed away from the socket (*i. e.*, tending to increase the thickness of the membrane).

The tangential displacement in the membrane has contravariant components

$$u^\alpha = x^\circ(x^\circ - h)\mathrm{B}^\alpha p/2\mu, \quad . \quad . \quad . \quad . \quad (2.3)$$

where x° is the normal distance from the socket of the point at which u^α is calculated.

† T_1, p. 450; T_2, p. 972.

3. *Equilibrium of a Tooth when Congestion is present.*

Let us suppose that some parts of the membrane are in normal condition, so that for those portions the equations of § 2 hold. We shall suppose that in the congested regions the state is hydrostatic, so that *in a congested region* $p=const.$ The volume of a congested region will not be the same as it was before congestion took place, on account of the presence of the additional blood responsible for the hydrostatic régime.

Let there be a tooth with root of given shape, with a membrane of given thickness h, in general variable. Let the atmospheric pressure be P, and let there be given regions of congestion on the root, in which the pressure has assigned values P_1, P_2, ... Two problems present themselves :

(i.) To find the displacement experienced by the tooth when subjected to given external forces.

(ii.) To find the external forces required to give the tooth an assigned displacement.

These two problems are not essentially distinct, because the force system is defined by six constants and the displacement is defined by six constants. Suppose the displacement assigned : we are then to solve (2.2) for p in the normal region, subject to conditions $p=P$, $p=P_1$, $p=P_2$, . . . on the boundaries of the normal region. We are then able to calculate the six components of the force system exerted on the tooth by the membrane and the atmosphere, and these six components reversed give the force system necessary to hold the tooth in equilibrium. If, on the other hand, the force system is assigned, we simply take general values for the six constants defining the displacement and proceed as before, ultimately determining these six constants by the condition that the force system required to produce this displacement is, in fact, the assigned system. We may regard the problem (ii.) as the standard problem.

There is, however, an interesting simpler problem, a special case of (ii.), namely :

(iii.) To find the external forces required to hold the tooth in normal position when regions of congestion are present.

This problem is simpler mathematically, because we have $\beta=0$, and hence (2.2) is no longer of the Poisson type, but of the Laplace type,

$$B^\alpha(h^3 B_\alpha p)=0. \quad . \quad . \quad . \quad . \quad . \quad (3.1)$$

The pressures in the congested regions will exceed atmospheric pressure. It is easy to see then that the pressure does not fall below atmospheric pressure throughout the whole of the membrane. For application of Green's Theorem to a region of the root enclosed by a contour C gives, by virtue of (3.1),

$$\int_C h^3 \frac{\partial p}{\partial n} ds=0, \quad . \quad . \quad . \quad . \quad . \quad (3.2)$$

where $\partial p/\partial n$ is the derivative of p along the outward normal to C. Hence, since h is of course positive, *p cannot be a maximum or a minimum except at a boundary of the normal region.* Its maximum will therefore be the greatest pressure occurring in a congested region, and this maximum will be attained in that region, whereas the minimum pressure will be atmospheric pressure, and it will occur at the edge of the membrane only.

4. *Points of Congestion.*

It need hardly be said that the major mathematical difficulties of the problem are involved in the solution of (2.2) or (3.1) under the assigned boundary conditions. It is obvious that our problem is closely related to problems in two-dimensional (or logarithmic) potential theory, generalized, however, by the presence of the variable factor h in the partial differential equation and by the fact that the region (the surface of the root) is not plane but curved. The boundary conditions are of the Dirichlet type, the value of p being assigned. Thus our problem is analogous to the electrostatic problem in which conductors are given assigned potentials. This electrostatic problem is more difficult than that in which some of the conductors are replaced by point-charges, and this suggests that we might profitably alter our elastic problem from one in which there are finite regions of constant pressure to one in which all the membrane is normal with the exception of isolated *points of congestion*, which we shall now proceed to define.

Let C be a closed curve drawn on the root (fig. 1), and let a flat curved cylinder Γ be erected on it by drawing

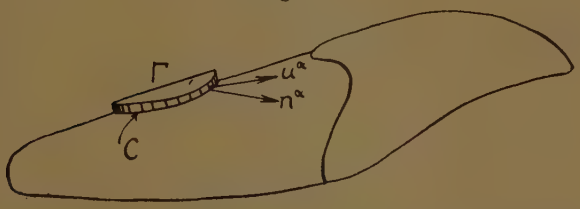
normals to the root along C, one face of the cylinder being on the root and the other on the socket. Let us compare the volume of Γ in the actual state with its volume in the unstressed state, when there are no congestions present and no forces (other than atmospheric pressure) applied to the tooth. The increase in volume is obviously

$$\delta V = \iint \beta \, dS + \iint u_n \, dS, \quad . \quad . \quad . \quad . \quad (4.1)$$

where the first integral is taken over the interior of C and the second integral is taken over the wall of Γ, u_n being the component normal to the wall of the displacement u^α; dS is the element of area in each case.

Now if n_α are the covariant components of a unit vector normal to C in the surface of the root, drawn

Fig. 1.



outwards from the interior of C, then $u_n = n_\alpha u^\alpha$, and the second integral in (4.1) may be written, by virtue of (2.3),

$$\iint u_n \, dS = \int_C n_\alpha \, ds \int_{x^3=0}^{h} u^\alpha \, dx^3 = -\frac{1}{12\mu} \int_C h^3 n_\alpha B^\alpha p \, ds$$

$$= -\frac{1}{12\mu} \int_C h^3 \frac{\partial p}{\partial n} \, ds, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.2)$$

where ds is the element of arc of C and $\partial p / \partial n$ the derivative of p along the outward normal to C. Thus

$$\delta V = \iint \beta \, dS - \frac{1}{12\mu} \int_C h^3 \frac{\partial p}{\partial n} \, ds. \quad . \quad . \quad (4.3)$$

If there is no congestion inside C, then $\delta V = 0$, and we may apply Green's Theorem on the surface of the root, obtaining

$$0 = \iint \beta \, dS - \frac{1}{12\mu} \iint B^\alpha (h^3 B_\alpha p) \, dS, \quad . \quad . \quad (4.4)$$

a result which simply verifies the equation (2.2).

Let us now suppose that congestion is present inside C. Actually the region of congestion will be a finite patch on the surface of the root, but we wish for reasons of mathematical convenience to treat rather of *points of congestion*, analogous to point-charges in electrostatics or point-sources in hydrodynamics. Accordingly we suppose that δV remains the same as C contracts to a point A. The first integral in (4.3) tends to zero; in order that the second may not tend to zero, we must suppose that $\partial p/\partial n$ tends to infinity as we approach A, in fact becoming infinite as $1/\rho$, if ρ denotes distance from A. We have then

$$\delta V = -\frac{h^3}{12\mu} \lim_{C \to A} \int_C \frac{\partial p}{\partial n} ds. \quad . \quad . \quad . \quad (4.5)$$

The point A *is then a point of congestion*; at it there occurs an increase in volume δV. It is convenient to define the *magnitude of the congestion* e at A by the formula

$$e = \frac{6\mu\delta V}{\pi h^3}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.6)$$

so that we have

$$e = -\frac{1}{2\pi} \lim_{C \to A} \int_C \frac{\partial p}{\partial n} ds. \quad . \quad . \quad . \quad . \quad (4.7)$$

We have just been concerned with a limiting process in which the curve C contracts to a point A. But there is another limit inherent in the problem, since h is infinitesimal. Remembering that μ is regarded as finite in our theory, it is evident from (4.6) that for a finite magnitude of congestion we must consider δV to be of the order of h^3.

The point of congestion is, of course, a mathematical fiction—an idealization of a small patch of congestion. The justification for its introduction may be based on the analogous process in electrostatics. If a small conductor carrying a charge e is replaced by a point-charge e, the potential at distant points is altered only by an amount that tends to zero as the size of the conductor tends to zero. It is only in the neighbourhood of the conductor that the approximation is bad.

We now change our boundary-value problem to the following:—*There being points of congestion in the membrane at* $A_1, A_2, \ldots,$ *the magnitudes of the congestions at*

these points being $e_1, e_2, \ldots$, *it is required to find* p *to satisfy the partial differential equation* (2.2), *to satisfy the condition* $p=\mathrm{P}$ (*atmospheric pressure*) *on the edge of the membrane, and to become infinite at* $\mathrm{A}_1, \mathrm{A}_2, \ldots$, *the principal part of* p *near* A_i *being* $-e_i \log \rho$, *where* ρ *is the distance from* A_i.

This last condition is merely an alternative expression of (4.7).

The problem (ii.) of § 3 in the presence of congestions involves the determination of p to satisfy (2.2), β being an assigned function of position over the surface of the root. This problem may be split into two parts :

(*a*) To find p when the displacement β is assigned and there are no congestions.

(*b*) To find p when $\beta=0$ and the congestions are present.

For, if we write

$$\bar{p}=p-\mathrm{P}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.8)$$

the excess over atmospheric pressure, the equation (2.2) reads

$$\mathrm{B}^{\alpha}(h^3 \mathrm{B}_{\alpha} \bar{p})=12\mu\beta. \quad . \quad . \quad . \quad . \quad (4.9)$$

Now if we find a solution $\bar{p}_1$ of this equation free from singularities, with $p_1=0$ on the edge of the membrane, and a solution $\bar{p}_2$ of

$$\mathrm{B}^{\alpha}(h^3 \mathrm{B}_{\alpha} \bar{p})=0, \quad . \quad . \quad . \quad . \quad (4.10)$$

with the required singularities and $p_2=0$ on the edge of the membrane, then $\bar{p}=p_1+\bar{p}_2$ is the solution of our problem.

The determination of p_1 is the problem discussed in the papers T_1 and T_2 ; we shall not pursue it further here. On the other hand, we shall turn our attention to the second problem, which is, in fact, the problem (iii.) of § 3, but with points of congestion instead of hydrostatic regions.

5. *The Case of a Membrane of Constant Thickness on a Conical Root.*

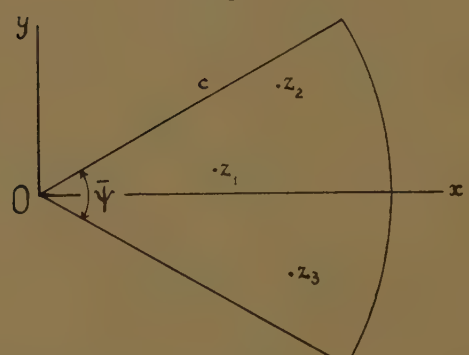
We shall now discuss the last-named problem, namely, the determination of the pressure and applied force system in the presence of points of congestion, but with no displacement of the tooth.

It is obvious that the remarks concerning maximum and minimum pressure made in § 3 apply here also. *The pressure throughout the membrane will exceed atmospheric pressure.*

Let us now suppose that the thickness h of the membrane is constant and that the root is conical (but not necessarily of revolution), so that the surface of the root may be developed on a plane. The equation (4.10) then reduces to

$$\Delta\bar{p}=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.1)$$

Fig. 2.



z-plane.

where Δ is the two-dimensional Laplacian operator. *Our problem is to find a function p harmonic over the surface of a cone, except at assigned points of congestion, with $p=0$ on a closed curve (the edge of the membrane) and becoming infinite with principal parts $-e_i \log \rho$ $(i=1, 2, \ldots)$ at the points of congestion.*

We shall confine our attention (as in T_2) to the case where the generators of the conical root are of constant length c from the vertex to the edge of the membrane. The development of the root in the plane is then a sector of a circle of radius c (fig. 2) with angle ψ (say)

Let us take in the plane development rectangular axes Ox, Oy, the origin O being at the vertex and Ox being the bisector of the angle of the sector. Let $z=x+iy$,

and let the developments of the points of congestion be at $z_1, z_2, \ldots$, the magnitudes being $e_1, e_2, \ldots$

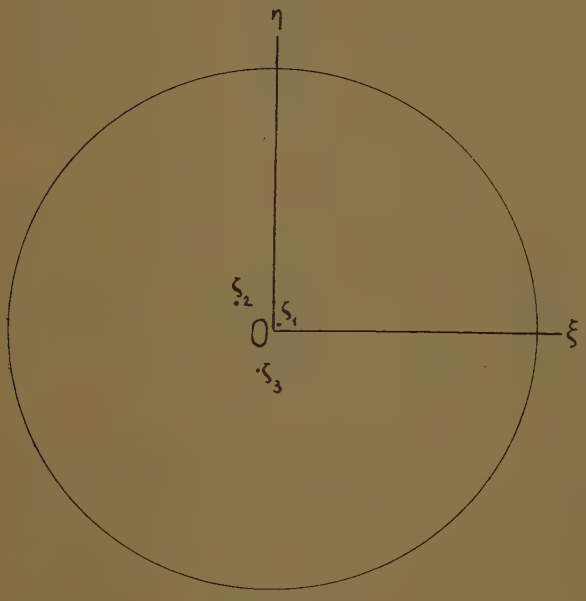
Let n (not necessarily an integer) be defined by

$$n=2\pi/\bar{\psi}, \qquad \bar{\psi}=2\pi/n, \quad . \quad . \quad . \quad . \quad (5.2)$$

and let us make a conformal representation of the z-plane on a ζ-plane ($\zeta=\xi+i\eta$) by the transformation

$$\zeta=(z/c)^n. \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.3)$$

Fig. 3.



ζ-plane.

This transforms the z-sector into the interior of the complete unit circle in the ζ-plane, and the points $z_1, z_2, \ldots$ into $\zeta_1, \zeta_2, \ldots$ Although ζ is a multiple-valued function of x, there will be no ambiguity in the representation, because we shall take

$$\arg\zeta=n\arg z, \qquad -\tfrac{1}{2}\bar{\psi}<\arg z\leqslant\tfrac{1}{2}\bar{\psi}. \quad . \quad (5.4)$$

The bounding radii of the z-sector are brought into coincidence on the negative ξ-axis, a fact which might

be an embarrassment were we really dealing with a sector, but is actually an advantage because the two bounding radii of the sector are in reality one single generator on the cone.

We now proceed in the usual way to get a harmonic function $\bar{p}$ in the ζ-plane with $p=0$ for $|\zeta|=1$, and with the required singularities. Denoting by q the harmonic function conjugate to p, and putting

$$w=\bar{p}+i\bar{q}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.5)$$

we have by the method of images

$$w=-\sum_j e_j \log \frac{\zeta-\zeta_j}{\zeta\zeta_j^*-1}, \quad . \quad . \quad . \quad . \quad (5.6)$$

where ζ_j^* is the conjugate complex number to ζ_j. Thus *the excess of the pressure over atmospheric pressure* ($p=p-\mathrm{P}$) *is given by the real part of*

$$\bar{p}+i\bar{q}=-\sum_j e_j \log \frac{(z^n-z_j^n)c^n}{z^n z_j^{*n}-c^{2n}}, \quad . \quad . \quad . \quad (5.7)$$

where z_j^* is the conjugate of z_j.

Let us now suppose that there is only one point of congestion of magnitude e at distance b from the vertex of the conical root. We may make our development so that this point lies on the bisector of the sector, that is, at $z=b$. Accordingly the excess of pressure is given by the real part of

$$\bar{p}+i\bar{q}=-e \log \frac{(z^n-b^n)c^n}{z^n b^n-c^{2n}}, \quad . \quad . \quad . \quad (5.8)$$

and so, with polar coordinates (r, ψ) in the plane of the development (so that the point of congestion is at $r=b$, $\psi=0$), we have

$$\begin{aligned} p &= -\frac{e}{2} \log \frac{[(r^n \cos n\psi - b^n)^2 + r^{2n} \sin^2 n\psi]c^{2n}}{(r^n b^n \cos n\psi - c^{2n})^2 + r^{2n} b^{2n} \sin^2 n\psi}, \\ &= \frac{e}{2} \log \frac{r^{2n} b^{2n} - 2r^n b^n c^{2n} \cos n\psi + c^{4n}}{(r^{2n} - 2r^n b^n \cos n\psi + b^{2n})c^{2n}} \quad . \quad . \quad . \quad (5.9) \end{aligned}$$

On the generator through the point of congestion ($\psi=0$) we have

$$\bar{p}=e \log \frac{c^{2n}-r^n b^n}{|r^n-b^n|c^n}, \quad . \quad . \quad . \quad . \quad (5.10)$$

becoming infinite for $r=b$ and vanishing for $r=c$. The excess of pressure at the vertex is

$$\bar{p}=en \log c/b. \quad . \quad . \quad . \quad . \quad . \quad (5.11)$$

Let us make a numerical estimate of this excess of pressure, using the approximate numerical values given in T_2 (pp. 993, 994). There we had

$$\left.\begin{aligned} n &= 2\pi/\bar{\psi} = 4{\cdot}428, \\ h &= 0{\cdot}24 \text{ mm.}, \\ c &= 16 \text{ mm.} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (5.12)$$

As regards μ, in T_1 and T_2 I used for numerical purposes the rigidity of rubber. However, some experimental work † on the membranes of calves and lambs showed that the rigidity of their membranes was considerably less. As a figure of the right order of magnitude we may take

$$\mu = 50 \text{ gm. mm.}^{-2} \quad . \quad . \quad . \quad . \quad . \quad (5.13)$$

(For rubber, $\mu = 163$ gm. mm.$^{-2}$.)

Let us assume that the point of congestion leads to an increase in volume

$$\delta V = 0{\cdot}001 \text{ mm.}^3 \quad . \quad . \quad . \quad . \quad . \quad (5.14)$$

For these figures the magnitude of the congestion is, by (4.6),

$$e = \frac{6 \times 50 \times 0{\cdot}001}{\pi(0{\cdot}24)^3} = 6{\cdot}9 \text{ gm. mm.}^{-2}. \quad . \quad (5.15)$$

Taking $b = \frac{1}{2}c$, we conclude from (5.11) that *a point of congestion producing a volume-increase of one-thousandth of a cubic millimetre at a point half-way up the root produces at the apex an excess over atmospheric pressure of amount*

$$\bar{p} = 21 \text{ gm. mm.}^{-2} = 30 \text{ lb. in.}^{-2}. \quad . \quad . \quad (5.16)$$

This result shows how very sensitive the tooth is to congestion in the membrane ; in the case just considered the pressure at the apex is raised to about three atmospheres by a very small increase in volume.

To obtain from (5.9) the pressure excess on the generator most remote from the point of congestion in the plane development, we are to put $\psi = \frac{1}{2}\bar{\psi} = \pi/n$, $n\psi = \pi$; thus we have

$$p = e \log \frac{r^n b^n + c^{2n}}{(r^n + b^n)c^n}. \quad . \quad . \quad . \quad . \quad (5.17)$$

† M. L. Dyment and J. L. Synge, 'Oral Health,' March 1935.

This excess is positive for all values of r from zero to c, in accordance with the general result mentioned earlier.

In the preceding discussion it has not been necessary to specify the *shape* of the conical root. This must, however, be done if we are to calculate the force system necessary to hold the tooth in its normal position.

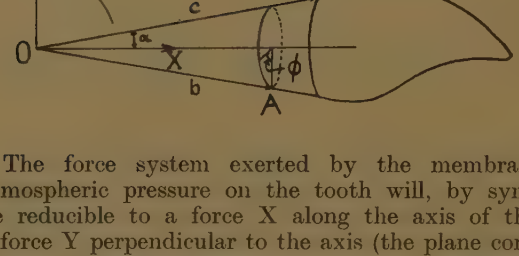
Let us suppose that the tooth is a *right circular cone of semi-angle* α. Then if ϕ is an azimuthal angle, measured from the diametral plane containing the point of congestion A, ϕ is connected with the polar angle ψ of the plane development by the equation

$$\psi = \phi \sin \alpha, \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.18)$$

and we have

$$2\pi/n = \overline{\psi} = 2\pi \sin \alpha, \qquad n = \operatorname{cosec} \alpha. \quad (5.19)$$

Fig. 4.



The force system exerted by the membrane and atmospheric pressure on the tooth will, by symmetry, be reducible to a force X along the axis of the root, a force Y perpendicular to the axis (the plane containing X and Y passing through the point of congestion A), and a couple N in the plane of X, Y. We shall take Y positive when directed away from the point of congestion A.

We have then (since atmospheric pressure all over the tooth contributes nothing to the force system)

$$\left.\begin{aligned} X &= \sin^2 \alpha \iint \bar{p} r \, dr \, d\phi, \\ Y &= \sin \alpha \cos \alpha \iint \bar{p} r \cos \phi \, dr \, d\phi, \\ N &= \sin \alpha \iint \bar{p} r^2 \cos \phi \, dr \, d\phi, \end{aligned}\right\}, \quad . \quad (5.20)$$

r being the distance from the vertex, the integrals being

taken over the range $0 \leqslant r \leqslant c$, $0 \leqslant \phi \leqslant 2\pi$. Since $n\psi = \phi$, we have for $\bar{p}$, by (5.9),

$$\bar{p} = \frac{e}{2} \log \frac{r^{2n} b^{2n} - 2r^n b^n c^{2n} \cos \phi + c^{4n}}{(r^{2n} - 2r^n b^n \cos \phi + b^{2n}) c^{2n}}, \quad . \quad (5.21)$$

$$n = \operatorname{cosec} \alpha.$$

The logarithmic singularity of $\bar{p}$ does not, of course, interfere with the existence of the integrals.

The determination of the force system is thus a mere matter of computation of these integrals for specific values of α and b/c.

Added in proof.—In this paper, and in those referred to, absolute incompressibility of the membrane is assumed. Dr. J. N. Goodier has recently pointed out to me that a small compressibility, comparable to that of water, gives rise to terms which cannot be neglected. I hope to prepare shortly a revised form of the theory, taking this small compressibility into account. Meanwhile quantitative results for the periodontal membrane, based on the hypothesis of absolute incompressibility, must be received with caution.

LXXXIX. *On Negative Resistance Equalizers for Telephone Transmission System.* *By* S. P. Chakravarti, *M.Sc. (London), D.I.C., A.M.I.E.E., University Lecturer, University of Calcutta* *.

Summary.

The paper relates to the application of "negative resistance" and "negative inductance" in the development of attenuation equalizer elements for telephone transmission systems. Non-reactive impedances of the type $Z = \pm\omega^2 \left(\frac{L^2}{r}\right)$ are obtained from combinations of positive and negative circuit constants. Attenuation, increasing gradually with frequency, is obtained by connecting an impedance of the type $Z = \omega^2 \left(\frac{L^2}{r}\right)$ in a

* Communicated by the Author.

circuit containing a source of A.C. power of internal resistance Z_0, a load resistance Z_0, and an additional adjustable resistance r_1. Attenuation, decreasing gradually with frequency, is obtained by connecting an an impedance of the type $Z = -\omega^2\left(\frac{L^2}{r}\right)$ in a circuit similar to that described above. Resistances r and r_1 control the form of characteristics.

An attempt has been made to show how far the gain of a transformer-coupled audio-frequency amplifier. increasing with frequency, can be corrected by the former characteristic and the attenuation of a cable circuit (for message telephone, or broadcast programme transmission), increasing with frequency, can be corrected by the latter characteristic. Measurement of phase-shift of the output voltage with respect to the input shows that delays for components of frequencies higher and lower than 1000 c.p.s. (mean speech frequency) relative to that of 1000 c.p.s. are very small to cause any phase distortion. The stability and the advantages of the present type of equalizer are discussed.

1. *Introduction.*

IF a telephone signal is to be transmitted without distortion over a transmission system, it is necessary that all component frequencies making up the signal be treated alike, from the point of view of transmission efficiency, and some frequencies may not be unduly emphasized to the detriment of others. Some portions of the transmission system traversed by the signal are incapable of uniform transmission. For example, wave-filters tend to discriminate against frequencies near the edges of their transmission bands, due to dissipation of energy in effective resistances of their elements and to reflexion losses at their terminal junctions. Loaded telephone lines also behave in a similar way. Unloaded telephone lines have an attenuation which increases gradually with frequency on account of the nature of capacity effects present. Some types of audio-frequency amplifiers, for instance, transformer-coupled audio-amplifiers, have a gain characteristic rising in general with frequency on account of transformer characteristics.

An equalizer, or a corrective network, is a device which compensates for variations in the transmission system by introducing variations in the opposite direction. It attenuates efficiently component frequencies in such a manner and to such an extent that they suffer or prosper alike in transmission. There are two principal types of attenuation characteristic which an equalizer may be called upon to correct—one, in which the attenuation gradually increases with frequency; and second, in which the attenuation gradually decreases with frequency. Unloaded cable and open-wire telephone lines have characteristics of the first type, whereas some of the audio-amplifiers have characteristics of the second type.

The present paper relates to formation of stable negative resistance and negative inductance, and their application in the development of equalizer elements for correcting both types of attenuation characteristics as stated above.

2. *Negative Circuit Constants.*

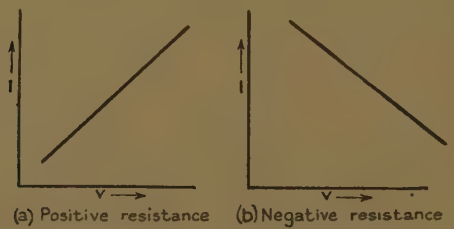
Various types of circuit offering "negative resistance" to an applied electromotive force have been discussed since 1918 by various workers, chief among them being Hull, Scott-Taggart, George Crisson, Mathes, and Dudley. Bartlett and Van der Pol have further suggested methods of obtaining "negative inductance" and "negative capacitance" from negative resistance. Negative circuit elements have also been considered, both with respect to steady and transient conditions, by Bartlett, Van der Pol, and Verman.

Negative resistance.—When a source of electromotive force is connected to the terminals A and B of an ordinary positive resistance, a current flows in at the terminal A connected to the positive pole of the source and out at the other terminal B. This direction of current flow is positive. Similarly, if a current be sent through the positive resistance in the same direction, A to B, a potential difference will appear across its terminals, A being at a higher potential than B.

There could be a device which has the property of keeping the ratio of voltage across the terminals to current flowing constant, but with the relative direction of

voltage and current opposite to that which a positive resistance would give. Such a device is called the "negative resistance" and contributes power to the circuit with which it is connected. There are two kinds of such devices—one, in which, if a definite voltage is applied across the terminals, a current flows in a direction opposite to that of the applied voltage; and second, in which, if a definite current is passed through the system, the potential difference across the terminals will be opposite in direction to that caused by a positive resistance. These two arrangements are entirely different and cannot be used interchangeably in a given circuit as in a case of a positive resistance. The voltage-current characteristics of a positive and a negative resistance are shown in fig. 1.

Fig. 1.

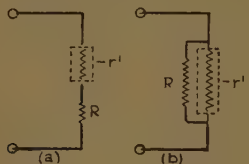


(a) Positive resistance (b) Negative resistance

There are many forms of negative resistances, namely, the electric arcs, the triodes and the tetrodes (screen-grid tubes) used as dynatrons, and a one-way distortionless amplifier of infinite input impedance in which input and output terminals are inter-connected. The primary difficulty is the *instability* of the devices. Thermionic vacuum-tubes are more stable than the electric arcs, but even the negative resistance obtained from commercial tubes has been found not to stay constant over a long period of time. The one-way amplifier of the type mentioned above is perhaps the most stable of all devices, but cost restricts the choice to thermionic vacuum-tubes used as dynatrons, specially when a transmission network requires the use of several negative resistance elements. Another difficulty is that it is not easy to vary the negative resistance value of a given vacuum-tube system beyond certain limits. It is possible, however, to vary the

negative resistance by connecting a positive resistance in series or parallel to it. Fig. 2 (*a*) shows how the absolute value of a given negative resistance $-r'$ can be reduced and fig. 2 (*b*) shows how it can be increased.

Fig. 2.



"Negative inductance" and "Negative capacitance".— After having obtained a stable negative resistance, it is possible to get a negative inductance and a negative capacitance. For negative inductance, the circuit

Fig. 3 (*a*).

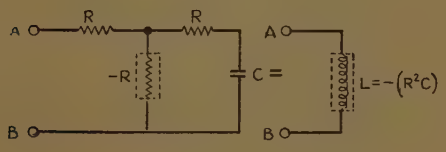
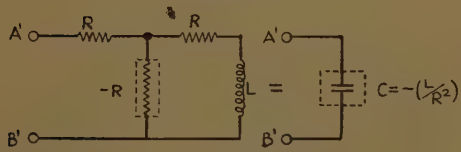


Fig. 3 (*b*).



connexion is shown in fig. 3 (*a*). The impedance $Z_{A-B}=R+Z'_p$, where Z'_p=the parallel impedance $=-R-j.\omega CR^2$. Hence

$$Z_{A-B}=j.\omega(-R^2C)=j\omega(L), \quad . \quad . \quad . \quad . \quad (1)$$

where $$L=-R^2C.$$

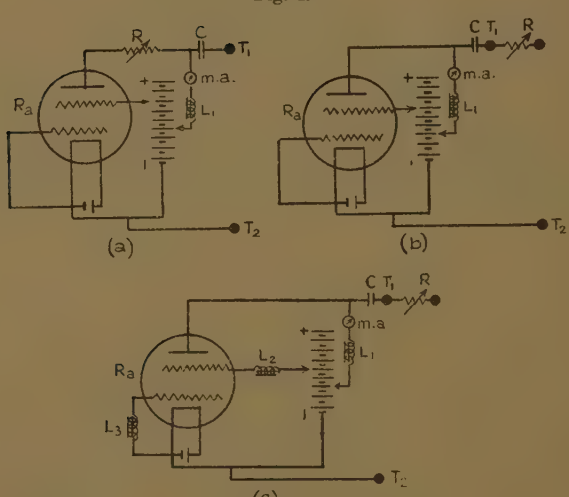
For negative capacitance, the circuit connexion is

shown in fig. 3(*b*). The impedanee $Z_{A'-B'}=R+Z''_p$, where Z''_p=the parallel impedance $=-R+j.\frac{R^2}{\omega L}$. Hence

$$Z_{A'-B'}=j.\frac{R^2}{\omega L}=-j.\frac{1}{\omega(-L/R^2)}=-j.\frac{1}{\omega C}. \quad . \quad (2)$$

where $$C=-\frac{L}{R^2}.$$

Fig. 4.



3. *Formation of Negative Resistance.*

The present section relates to formation of stable negative resistances which form the basis for all other types of negative circuit constants. The internal resistance of a screen-grid tube under secondary emission condition in series with a variable positive resistance R was employed to obtain negative resistances of different values. Several screen-grid tubes manufactured by different firms were studied for their secondary emission. Mazda SG 215 was found to be most suitable for the purpose. Figs 4 (*a*), 4 (*b*). and 4 (*c*) show three different circuit arrangements.

In fig. 4 (*a*) the variable positive resistance R is connected directly in the anode circuit of the tube. Since the value of R has to be quite large of the order of R_a (the internal resistance of the tube) to obtain a net resistance of, say, -1000 ohms, the d.c. voltage drop across R is considerable, and the anode voltage V_a has therefore to be increased to keep the working point on the negative portion of anode voltage—anode current characteristic. C is a blocking condenser (of large value) to prevent the direct current from circulating in the main circuit. L_1 is a choke to prevent the high-frequency current in the main circuit from circulating in the parallel battery path.

In fig. 4 (*b*) the variable positive resistance is connected externally to the negative resistance circuit and does not disturb the working point as in the previous case. In fig. 4 (*c*) additional chokes L_2 and L_3 have been inserted to prevent high-frequency currents from taking parallel paths through screen-grid and control-grid circuits. However, it has been found experimentally that there is no appreciable difference in measurements whether fig. 4 (*b*) or fig. 4 (*c*) circuit is adopted. Arrangement in fig. 4 (*b*) has therefore been employed in the present experiments.

The stability of negative resistance obtained from the screen-grid tube depends upon (*a*) the constancy of filament current, anode current, anode voltage, and screen-grid voltage ; and (*b*) the condition of the surface emitting secondary electrons. Various currents and voltages in the tube circuit were maintained constant throughout the experiment. Anode voltage (H.T. voltage minus d.c. drop in the choke)—anode current characteristic of the tube is shown in fig. 5. An overall A.C. voltage—A.C. current characteristic for the whole arrangement across T_1–T_2 was taken at a frequency of 1000 cycles/second and found to agree closely with the static characteristic in fig. 5.

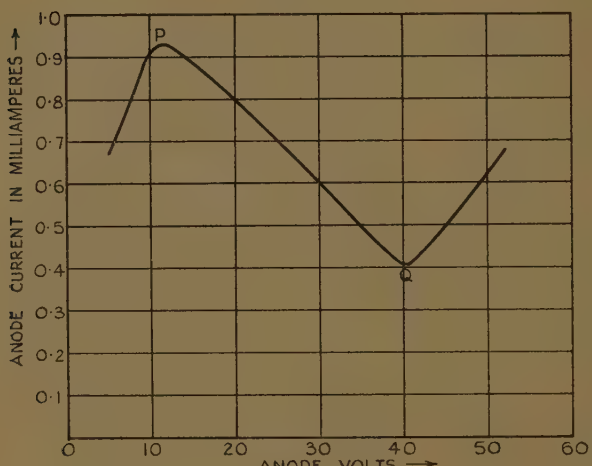
The portion PQ of the anode voltage—anode current characteristic was used for obtaining the negative resistance, the magnitude of which is given by

$$R_a = \frac{\delta V_a}{\delta I_a} = -50{,}000 \text{ ohms},$$

which is constant whichever point on PQ was chosen

as the working point. Since the blocking condenser is one of large capacitance, the resultant negative impedance could be taken without appreciable error as a non-reactive resistance of value $-R_a + R = -R'$ ohms.

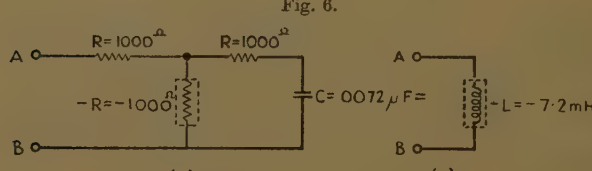
Fig. 5.



4. *Equalizer for correcting Attenuation Characteristic gradually falling with Frequency.*

A negative resistance of -1000 ohms was first obtained by means of the circuit discussed in the last Section.

Fig. 6.



To obtain a " negative inductance " of $-7{\cdot}2$ millihenries, the circuit connexion was made as shown in fig. 6 (*a*) which is equivalent to circuit in fig 6 (*b*). The " equalizing

element " shown in fig. 7 (*a*) was formed of two positive inductance coils (of negligible resistance) each 7·2 millihenries, negative inductance of −7·2 millihenries, as shown in fig. 6 (*a*), and an adjustable positive resistance (*r*).

If Z_{C-D} be the impedance looking into C—D to terminals at the angular frequency ω, then $Z_{C-D}=j\omega L+Z_p$, where the parallel impedance $Z_p=\frac{\omega^2L^2}{r}-j\omega L$. Hence

$$Z_{C-D}=\omega^2\left(\frac{L^2}{r}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Fig. 7.

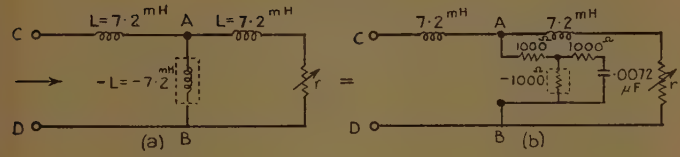
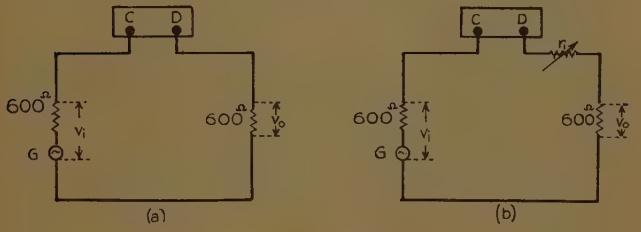


Fig. 8.



Z_{C-D} is non-reactive and directly proportional to the square of the angular frequency (hence, of frequency). With usual positive circuit constants, only impedances directly or inversely proportional to the first power of the impressed frequency, such as $j\omega L$ and $-j/\omega C$, are available.

Terminals C–D were then connected in a circuit containing a source of A.C. power of variable frequency and internal impedance of 600 ohms (non-reactive) and an external (load) resistance of 600 ohms, as shown in fig. 8(*a*). The voltage across the external resistance (V_0) decreased

with the rise of frequency and the attenuation of the system in decibels given by

$$\alpha = 20 \log_{10} \left| \frac{V_i}{V_0} \right|,$$

where V_i=input voltage (generator voltage minus voltage drop in internal resistance) and V_0=output voltage, increased gradually with frequency.

Fig. 8 (*b*) shows an additional non-reactive resistance r_1 in the circuit. Its effect was to vary the rate of increase of attenuation with frequency. Two control resistances r and r_1 enabled one to bring about a desired equalizer characteristic which would correct a given transmission characteristic. Table I. shows the variation of attenuation with frequency (both calculated and measured) for two values of "r," 2 and 10 ohms. For each value of r, two values of "r_1" (0 and 600 ohms) have been taken. Audio-frequency power was obtained from Standard Telephone and Cables Coy.'s audio-oscillator (type 74300) and voltages were measured by a calibrated thermionic voltmeter. An agreement between measured and calculated values of attenuation confirmed that the equalizer element formed of positive and negative circuit constants was sufficiently stable.

Effect of reducing the value of r from 10 ohms to 2 ohms was to make the characteristic steeper, whereas that of altering r_1 (keeping r constant) was to give characteristics with initial loss at low frequencies.

The relation between output and input voltages of the equalizer network can now be calculated. If E be the e.m.f. of the generator and I the resulting current, then from fig. 8 (*b*),

$$I = \frac{E}{2Z_0 + r_1 + \omega^2\left(\frac{L^2}{r}\right)}, \quad \ldots \quad (4)$$

where Z_0 is the value of internal impedance of the generator and also of load resistance

$$V_i = E - IZ_0 \quad \text{and} \quad V_0 = IZ_0.$$

$$\therefore \quad \frac{V_i}{V_0} = \frac{E - IZ_0}{IZ_0} = \frac{E}{IZ_0} - 1 = 1 + \frac{1}{Z_0}\left(\frac{\omega^2 L^2}{r} + r_1\right). \quad (5)$$

TABLE I.

f, in c.p.s.	ω, in r.p.s.	$r=2$ ohms.					$r=10$ ohms.				
		α in decibels.					α in decibels.				
		Z_{C-D} ohms.	$r_1=0$ ohms.		$r_1=600$ ohms.		Z_{C-D} ohms.	$r_1=0$ ohms.		$r_1=600$ ohms.	
			Calculated.	Measured.	Calculated.	Measured.		Calculated.	Measured.	Calculated.	Measured.
				(1)		(2)			(3)		(4)
100	628	9·3	0·35	0·45	6·2	6·3	2·0	0·0	·17	6·2	6·3
500	3140	250·0	3·20	3·50	7·8	7·5	50·0	0·7	·80	6·6	6·5
800	5024	650·0	6·55	6·30	10·0	9·5	130·0	1·7	1·70	7·0	6·9
1000	6280	998·4	8·60	8·20	11·5	11·0	200·0	2·4	2·30	7·4	7·2
1500	9420	2288·0	13·80	13·10	15·1	14·9	458·0	5·0	4·80	9·1	9·0
2000	12,560	4056·0	18·20	17·90	18·9	18·1	811·0	7·8	7·50	10·5	10·0
3000	18,840	9178·0	23·90	22·90	24·9	23·9	1836·0	12·6	12·0	14·4	14·0

For a transformer-coupled audio-amplifier, the magnification M is given, as usual, by

$$M = \frac{V_0}{V_i}$$

$$= \mu \sqrt{\frac{K^2 L_1 L_2}{\left[L_1\left\{1-(1-K)\left(\frac{f}{f_0}\right)^2\right\}\right]^2 + \left[\frac{R_a}{f}\left\{1-\left(\frac{f}{f_0}\right)^2\right\}\right]^2}},$$

where L_1=inductance of transformer primary; L_2 =inductance of transformer secondary; K=coefficient of coupling; μ=amplification factor of the tube; R_a=internal impedance of tube; and f_0=resonant frequency of transformer circuit.

In a typical case, $\mu=12{\cdot}7$; $R_a=8700^{\Omega}$; $L_1=80^{H}$; $L_2=920^{H}$; $K=0{\cdot}99$; $f_0=3540$ f.p.s. For values of f between 110 c.p.s. and 3540 c p.s., $(1-K)\left(\frac{f}{f_0}\right)^2$ can be neglected and $1-\left(\frac{f}{f_0}\right)^2$ is a fraction decreasing in value as f approaches f_0.

Hence

$$\frac{V_0}{V_i} \simeq \mu \sqrt{\frac{K^2 L_1 L_2}{L_1^2 + \frac{R_a^2}{f^2}\left(\frac{f_0^2-f^2}{f_0^2}\right)^2}}$$

$$\sim K'\left[1-\frac{1}{2}\cdot\frac{R_a^2}{L_1^2 f^2}\left(\frac{f_0^2-f^2}{f_0^2}\right)^2\right], \quad . \quad (6)$$

where

$$K' = \mu K \sqrt{\frac{L_1}{L_2}}.$$

As frequency (f) increases, magnification V_0/V_i of the amplifier increases and approaches K′. Fig. 9 shows how far the gain-frequency characteristic of a transformer coupled audio-amplifier may be corrected by various equalizer characteristics obtained from measured values. Table II. shows gains at three frequencies before and after equalization.

Fig. 9.

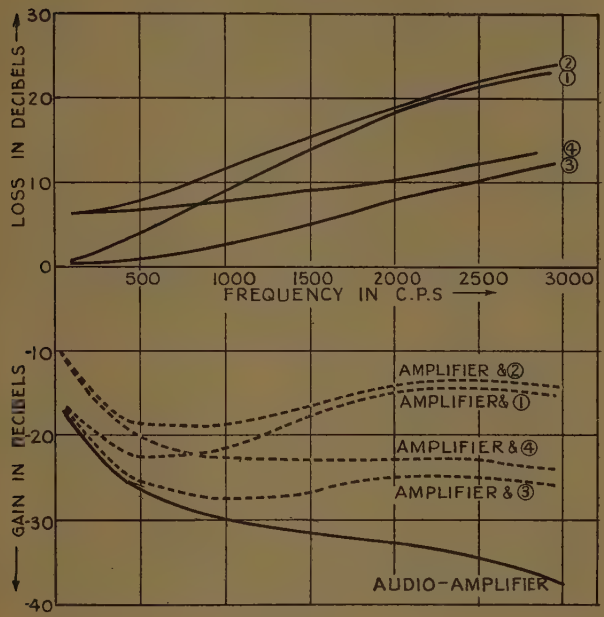


TABLE II.

	200 c.p.s.	1000 c.p.s.	3000 c.p.s.
Before equalization .	−19·5 db.	−30·0 db.	−38·0 db.
After equalization by characteristic :—			
No. (1)	−18·5 „	−22·0 „	−15·5 „
„ (2)	−13·0 „	−19·0 „	−14·5 „
„ (3)	−19·0 „	−27·7 „	−26·0 „
„ (4)	−13·5 „	−22·5 „	−24·0 „

5. *Equalizer for correcting Attenuation Characteristic gradually rising with Frequency.*

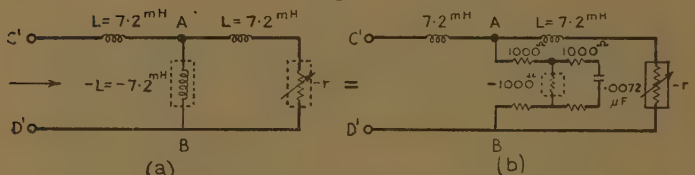
The equalizing element shown in fig. 10 (*a*) was formed of two positive inductance coils (of negligible resistance)

each 7·2 millihenries, negative inductance of −7·2 millihenries, as shown in fig. 6 (*a*), and a negative resistance whose value was adjustable.

If $Z_{C'-D'}$ be the impedance looking into C′−D′ terminals at the angular frequency, ω, then

$$Z_{C'-D'} = -\omega^2\left(\frac{L^2}{r}\right) \quad . \quad . \quad . \quad . \quad . \quad (7)$$

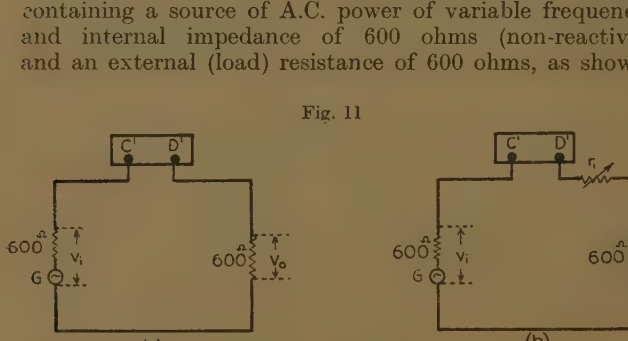
Fig. 10.



by calculations similar to those in the previous section. This impedance is a non-reactive negative resistance proportional to ω^2.

Terminals C′–D′ were then connected in a circuit containing a source of A.C. power of variable frequency and internal impedance of 600 ohms (non-reactive) and an external (load) resistance of 600 ohms, as shown

Fig. 11



in fig. 11 (*a*). The voltage across the external resistance (V_0) increased with rise of frequency and the attenuation of the system in decibels given by

$$\alpha = 20 \log_{10}\left|\frac{V_i}{V_0}\right|$$

decreased gradually with increase of frequency.

The effect of an additional resistance r_1 was to vary the rate of decrease of attenuation with frequency. Two control resistances, r and r_1, enabled one to obtain a desired characteristic which might correct a given transmission characteristic.

Table III. shows the attenuation of the system at various frequencies (both calculated and measured) for two values of negative resistance r, 10, and 100 ohms. For each value of r, two values of r_1 (300 and 600 ohms) have been taken. An agreement between measured and calculated values of attenuation confirmed that equalizer element formed of positive and negative circuit constants was sufficiently stable.

Effect of reducing the magnitude of the negative resistance r from 100 to 10 ohms was to decrease the attenuation at higher frequencies, keeping attenuation at the lowest frequency the same. Effect of reducing r_1 (keeping r constant) was to alter the initial attenuation.

The relation between input and output voltages of the equalizer net-work is given by

$$\frac{V_i}{V_0}=1+\frac{r_1}{Z_0}-\frac{\omega^2}{Z_0}\left(\frac{L^2}{r}\right)=1-\frac{1}{Z_0}\left[\frac{\omega^2L^2}{r}-r_1\right].$$

For values of r_1 from 300 to 600 ohms and for frequency ranges considered above, namely,

100–2000 c.p.s. for characteristics nos. (1) and (2), and

100–6000 c.p.s. for characteristics nos. (3) and (4),

$$\left|\frac{\omega^2L^2}{r}-r_1\right|<|Z_0|.$$

Therefore, by exponential theorem,

$$\frac{V_i}{V_0}=e^{-\frac{1}{Z_0}\left(\frac{\omega^2L^2}{r}-r_1\right)}, \text{ approximately,}$$

or
$$V_0=V_ie^{-\frac{1}{Z_0}\left(\frac{\omega^2L^2}{r}-r_1\right)}=V_ie^{-a}, \quad . \quad . \quad . \quad . \quad (8)$$

where
$$a=\frac{1}{Z_0}\left(\frac{\omega^2L^2}{r}-r_1\right).$$

In case of a telephone circuit of length "l," terminated in its "characteristic impedance,"

$$V_0=V_ie^{-\alpha l}=V_ie^{-A}, \quad . \quad . \quad . \quad . \quad . \quad (9)$$

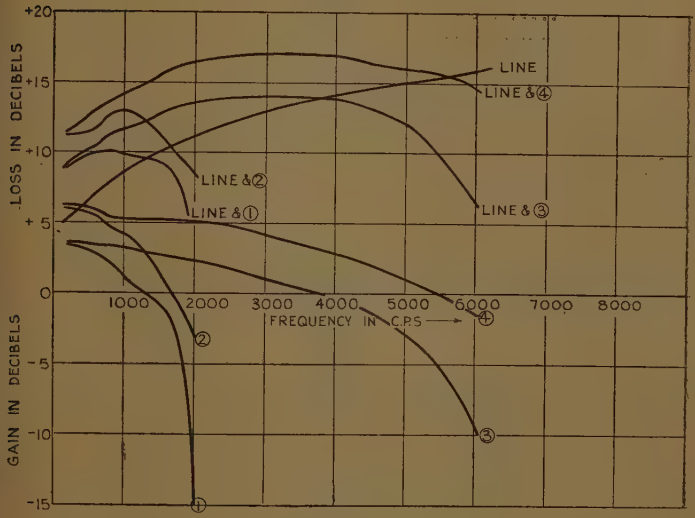
TABLE III.

f, in c.p.s.	ω, in r.p.s.	$r = -10$ ohms.					$r = -100$ ohms.				
		$Z_{C'-D'}$ ohms.	α in decibels.				$Z_{C'-D'}$ ohms.	α in decibels.			
			$r_1 = 300$ ohms.		$r_1 = 600$ ohms.			$r_1 = 300$ ohms.		$r_1 = 600$ ohms.	
			Calculated.	Measured.	Calculated.	Measured.		Calculated.	Measured.	Calculated.	Measured.
				(1)		(2)			(3)		(4)
100.....	628	— 2·0	3·5	3·8	6·0	6·3	— 0·2	3·6	3·8	6·1	6·4
500.....	3140	— 50·0	3·2	3·3	5·6	5·8	— 5·0	3·5	3·6	6·0	6·1
800.....	5024	—130·0	2·4	2·3	5·0	4·8	— 13·0	3·4	3·5	5·9	5·5
1000.....	6280	—200·0	1·4	1·2	4·6	4·5	— 20·0	3·3	3·2	5·85	5·5
1500.....	9420	—458·0	— 0·64	— 0·5	1·8	1·5	— 46·0	3·0	2·8	5·7	5·3
2000.....	12,560	—811·0	—16·5	—15·2	—3·74	—2·9	— 81·1	2·7	2·5	5·45	5·1
3000.....	18,840	..	..	..	..	..	—184	1·5	1·2	4·6	4·1
4000.....	25,120	..	..	.	..	..	—325	— 0·34	— 0·25	3·2	2·9
5000.....	31,400	..	..	..	..	..	—513	— 3·8	— 3·0	1·4	1·0
6000.....	37,680	..	..	..	..	..	—739	—11·0	—10·0	—2·1	—1·6

where α=attenuation constant and A=total attenuation of the circuit.

"a" depends upon ω^2 and magnitudes of r and r_1, whereas A depends upon l and α. In case of cable circuits $\alpha \simeq \sqrt{\frac{CR\omega}{2}}$, where C and R are capacitance and resistance per loop mile respectively.

Fig. 12.



Hence proper adjustment of values of r and r_1 is necessary to bring about a suitable equalizer characteristic.

Fig. 12 shows how far the attenuation of a cable circuit, 13 miles long, of 1·4 mm. diameter, annealed copper conductors employed for broadcast programme transmission may be corrected by various equalizer characteristics obtained from measured values. Table IV. shows the transmission losses at four frequencies before and after equalization. Equalizer characteristic No. (1) is evidently unsuitable for the purpose.

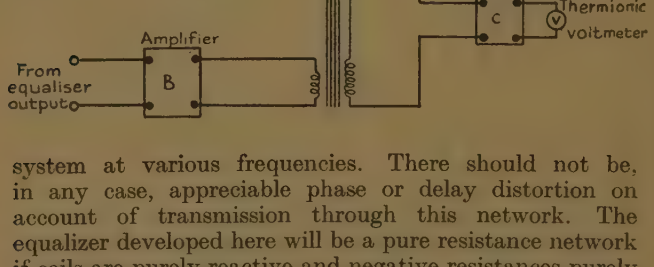
TABLE IV.

	100 c.p.s.	1000 c.p.s.	2000 c.p.s.	5000 c.p.s.
Before equalization ..	4·8 db.	8·7 db.	11·3 db.	15·2 db.
After equalization by characteristic :—				
No. (1)	8·9 „	10·0 „	−11·3 „	—
„ (2)	11·4 „	13·0 „	8·5 „	—
„ (3)	8·9 „	11·8 „	13·7 „	12·1 „
„ (4)	11·4 „	14·2 „	16·4 „	16·0 „

6. *Phase-shift in Negative Resistance Attenuation Equalizers.*

An attenuation equalizer is designed to correct the inequalities of gain or attenuation of a transmission

Fig. 13.

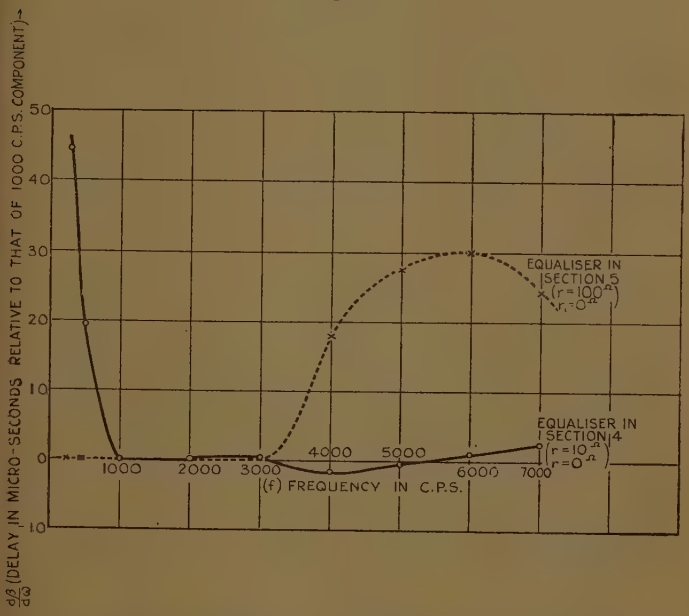


system at various frequencies. There should not be, in any case, appreciable phase or delay distortion on account of transmission through this network. The equalizer developed here will be a pure resistance network if coils are purely reactive and negative resistances purely non-reactive, and will give no phase-shift on account of transmission through it. An attempt was made to verify this by measuring the phase-shift of the equalizer output relative to its input. The method adopted was similar to that employed by Messrs. Hinton, Rendall, and White, in their paper on "Phase Relations in Unbalanced Two-way Telephone Repeaters" *. The circuit is shown in fig. 13.

* 'Electrical Communication,' viii. (July 1929).

Amplifiers A, B, and C, of which A and B are exactly identical, had very high input impedance and were efficient at all frequencies at which it was desired to measure the phase difference. T was a transformer with a primary winding consisting of two identical halves, each of which was connected to the output of A and B amplifier respectively. The secondary winding was connected through the amplifier C to the output measuring voltmeter.

Fig. 14.



Two potential differences whose phase difference was required, were applied to the inputs of A and B amplifiers respectively, first separately and then together. Three readings of voltage were thus obtained. From these three readings, the ratio of magnitudes of the potential difference applied to A and B and their phase difference were calculated.

If $V_i = M'E_0 \sin \omega t$ and $V_0 = E_0 \sin(\omega t \pm \beta)$ be the potential differences at the input and the output of the

equalizer system respectively, then the phase-shift between them is given by

$$\beta=\cos^{-1}\left[\frac{M'}{2}\left\{\left(\frac{V_{12}}{V_1}\right)^2-1-\frac{1}{M'^2}\right\}\right], \quad . \quad (10)$$

where $$M'=\left|\frac{V_1}{V_2}\right|;$$

V_1=R.M.S. value of voltage when input to A amplifier is only applied ; V_2=R.M.S. value of voltage when input to B amplifier is only applied, and V_{12}=R.M.S. value of voltage when both potential differences are applied simultaneously to the inputs of A and B amplifiers.

β, converted into radians, was plotted against ω. The value of $\frac{d\beta}{d\omega}$ gave the delay in micro-seconds. Fig. 14 shows the delay at different frequencies relative to that at 1000 cycles/sec. plotted against frequency. It is evident that relative delays of the order obtained by measurement are unable to cause any phase distortion.

7. *Conclusion.*

The present experiments have shown the possibility of developing equalizer elements from negative circuit constants to correct the attenuation characteristic of a telephone circuit or an audio-amplifier. "Negative resistance" and "negative inductance" which form impedances of the type $Z=\pm\omega^2\left(\frac{L^2}{r}\right)$ have been found to be sufficiently stable for practical purposes, though a good deal of work could be done in the development of stable negative resistances that would remain constant over a long period of time and be applicable to A.C. circuits.

Series circuits containing all positive or all negative elements are known to give rise to transients that die out exponentially after a short interval of time. It has been shown theoretically by Verman that series circuits containing one positive and one negative element produce constantly increasing transients. It is impossible for the currents to increase indefinitely. The rising currents can occur only for a certain interval of time. The negative elements would cease to function as such after a certain

limiting value of current is reached, since negative constants are only obtainable within definite current limits.

Three combinations of parallel impedance arise in the present paper for consideration with reference to transients. In fig. 6 (*a*), positive resistance (R) and positive capacitance (C) in series are parallel to the negative resistance (−R). In fig. 7 (*a*), positive inductance (L) and positive resistance (*r*) in series are parallel to the negative inductance (−L). In fig. 10 (*a*), positive inductance (L) and negative resistance (−*r*) in series are parallel to the negative inductance (−L). In first and second cases, transients will die out exponentially after a short interval of time, but in the third case it could be expected that increasing transients would be produced. The present experiments in which calculated and measured values of attenuation show close agreement support the view that transients do not increase indefinitely so as to stop functioning of the negative elements.

The present type of equalizer has several advantages over the usual type, namely;—

(1) Calculations involved in the design of equalizing element are much simpler. It is easier to bring about a desired correcting characteristic by adjusting values of r and r_1.

(2) It is a network of pure non-reactive elements and consequently introduces no phase distortion.

(3) It does not introduce much transmission loss in the transmission system.

On the other hand, it may be more expensive on account of the use of thermionic vacuum tubes and associated batteries. When equalizer element has to be inserted at a telephone repeater station or with a high quality amplifier system, batteries can be eliminated by making use of existing low-tension and high-tension batteries.

In conclusion, the author desires to thank Prof. P. N. Ghosh, Sc D, F Inst.P., of the Department of Applied Physics, University of Calcutta, for giving him all facilities when the work was being carried on in the electrical engineering laboratories of the department.

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XC. *Operational Forms for Bessel and Struve Functions.* *By* N. W. McLachlan, *D.Sc.*, *and* A. L. Meyers, *B.Sc.*, *A.M.I.E.E.* *

1. *Introduction.*

THE operational forms given in this paper come under two categories: (*a*) those obtained from

$$f(t)=\frac{1}{2\pi i}\int_{\mathrm{L}} e^{zt}\phi(z)\frac{dz}{z}, \quad . \quad . \quad . \quad . \quad (1)$$

the operational form being $\phi(p)$; (*b*) those obtained from the Laplace transform

$$\phi(p)=p\int_0^\infty e^{-pt}f(t)\,dt. \quad . \quad . \quad . \quad . \quad (2)$$

In (1) L is a contour illustrated in fig. 1 of a former paper †. The familiar contour $-\infty, (0+), -\infty$ is also used, following H. V. Lowry ‡, and will be designated L′. These contours are equivalent under conditions specified in the former paper.

* Communicated by the Authors.
† Phil. Mag. xxiii. p. 762 (1937).
‡ Phil. Mag. xiii. p. 1033 (1932).

In case (a) when $\phi(p)$ is expanded there may be positive integral powers of p, but these yield zero on being interpreted, since the corresponding integral

$$\frac{1}{2\pi i}\int_{\mathrm{L}} e^{zt}z^{n-1}\,dz=0, \quad n \geqslant 1.$$

$\mathrm{J}_0(at)/t$ in (10) below is a case in point. For (b) the contour $c-i\infty$ to $c+i\infty$ can be implied by virtue of the Mellin-Fourier theorem, provided

$$\left|\ \phi(p)\ /\ p\ \right| \underset{|p|\to\infty}{\to} 0, \quad -\tfrac{1}{2}\pi < \text{phase}\, p < \tfrac{1}{2}\pi.$$

Some of the operational forms in § 3 (derived from (2)) have terms of the type p^n, n integral $\geqslant 1$, but these are neutralized by identical terms of opposite sign. Such terms cannot be interpreted with reference to (1), using the contour $c-i\infty$ to $c+i\infty$, since the integral is oscillatory.

2. *Bessel Functions.*

To find the operational forms of various Bessel functions using the contours L or L′, it is merely necessary to select $\phi(z)$ from corresponding integrals in the former paper, which involve t in the exponential term e^{zt} only, and to replace z by p. $\phi(z)$ is, of course, a function of z alone and does not include t. Using the abbreviations

$$\mathrm{P}=(p+\sqrt{p^2+a^2}),\ \mathrm{Q}=(p+\sqrt{p^2-a^2}),\ \mathrm{S}=(p+\sqrt{p^2-i}),$$

and taking n for integral, ν for non-integral orders, the following operational forms are obtained:

$$\mathrm{Y}_\nu(at)\doteqdot p[a^\nu \mathrm{P}^{-\nu}\cos\nu\pi - a^{-\nu}\mathrm{P}^\nu]/\sin\nu\pi\sqrt{p^2+a^2}, \quad . \quad (3)$$

$$\mathrm{Y}_n(at)\doteqdot -p[a^n\mathrm{P}^{-n}+(-1)^n a^{-n}\mathrm{P}^n]\log(\mathrm{P}/a)/\pi\sqrt{p^2+a^2}, \quad . \quad . \quad . \quad (4)$$

$$\mathrm{Y}_0(at)\doteqdot -2p\log(\mathrm{P}/a)/\pi\sqrt{p^2+a^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$\mathrm{K}_\nu(at)\doteqdot \pi p[a^{-\nu}\mathrm{Q}^\nu - a^\nu\mathrm{Q}^{-\nu}]/2\sin\nu\pi\sqrt{p^2-a^2}, \quad . \quad . \quad (6)$$

$$\mathrm{K}_n(at)\doteqdot (-1)^n p[a^{-n}\mathrm{Q}^n + a^n\mathrm{Q}^{-n}]\log(\mathrm{Q}/a)/2\sqrt{p^2-a^2}, \quad (7)$$

$$\mathrm{K}_0(at)\doteqdot p\log(\mathrm{Q}/a)/\sqrt{p^2-a^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

$$\nu\mathrm{J}_\nu(at)/a\ t\doteqdot p/\mathrm{P}^\nu, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$\mathrm{J}_0(at)/t\doteqdot -p\log\mathrm{P}, \quad , \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

$$[(t-c)/(t+c)]^{\nu/2}\mathrm{J}_\nu(a\sqrt{t^2-c^2}) \doteq a^\nu p e^{-c\sqrt{p^2+a^2}}/\mathrm{P}^\nu\sqrt{p^2+a^2}, \quad t>c, \quad . \quad . \quad . \quad (11)$$

$$\nu \mathrm{I}_\nu(at)/a^\nu t \doteq p/\mathrm{Q}^\nu, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

$$\mathrm{I}_\nu(at)/t \doteq -p \log \mathrm{Q}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

$$[(t-c)/(t+c)]^{\nu/2}\mathrm{I}_\nu(a\sqrt{t^2-c^2}) \doteq a^\nu p e^{-c\sqrt{p^2-a^2}}/\mathrm{Q}^\nu\sqrt{p^2-a^2}, \quad t>c, \quad . \quad . \quad (14)$$

$$\nu \mathrm{Y}^\nu(at)/t \doteq p[a^{-\nu}\mathrm{P}^\nu + a^\nu \mathrm{P}^{-\nu} \cos \nu\pi]/\sin \nu\pi, \quad . \quad . \quad . \quad . \quad (15)$$

$$n\mathrm{Y}_n(at)/t \doteq p[(-1)^n a^{-n}\mathrm{P}^n - a^n \mathrm{P}^{-n}] \log (\mathrm{P}/a)/\pi, \quad . \quad . \quad (16)$$

$$\mathrm{Y}_0(at)/t \doteq p\{\log (\mathrm{P}/a)\}^2/\pi, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

$$\nu \mathrm{K}_\nu(at)/t \doteq -p\pi[a^{-\nu}\mathrm{Q}^\nu + a^\nu \mathrm{Q}^{-\nu}]/2 \sin \nu\pi, \quad . \quad . \quad . \quad . \quad (18)$$

$$n\mathrm{K}_n(at)/t \doteq (-1)^{n+1} p[a^{-n}\mathrm{Q}^n - a^n \mathrm{Q}^{-n}] \log (\mathrm{Q}/a)/2, \quad . \quad . \quad (19)$$

$$\mathrm{K}_0(at)/t \doteq -p\{\log (\mathrm{Q}/a)\}^2/2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

$$i^{-3\nu/2}(\mathrm{ber}_\nu t + i\,\mathrm{bei}_\nu t) \doteq p/\mathrm{S}^\nu\sqrt{p^2-i}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

$$\mathrm{ber}\, t \doteq \{(1+\sqrt{1+1/p^4})/2(1+1/p^4)\}^{1/2}, \quad . \quad . \quad . \quad . \quad (22)$$

$$\mathrm{bei}\, t \doteq \{(-1+\sqrt{1+1/p^4})/2(1+1/p^4)\}^{1/2}, \quad . \quad . \quad . \quad (23)$$

$$2^\nu t^{\nu/2}\,\mathrm{ber}_\nu\, t^{1/2} \doteq \frac{1}{p^\nu}\cos[(1/4p)+(3\nu\pi/4)], \quad . \quad . \quad . \quad . \quad . \quad (24)$$

$$2^\nu t^\nu\,\mathrm{bei}_\nu\, t^\nu \doteq \frac{1}{p^\nu}\sin[(1/4p)+(3\nu\pi/4)], \quad . \quad . \quad . \quad . \quad . \quad (25)$$

$$i^{-3\nu/2}\nu(\mathrm{ber}_\nu t + i\,\mathrm{bei}_\nu t)/t \doteq p/\mathrm{S}^\nu, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

$$(\mathrm{ber}\, t + i\,\mathrm{bei}\, t)/t \doteq -p \log \mathrm{S}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

$$\mathrm{ber}_\nu^2 t^{1/2} + \mathrm{bei}_\nu^2 t^{1/2} \doteq \mathrm{I}_\nu(1/2p), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

$$i^{3\nu/2}(\mathrm{ker}_\nu t + i\,\mathrm{kei}_\nu t) \doteq p\pi[\mathrm{S}^\nu - i^\nu \mathrm{S}^{-\nu}]/2 \sin \nu\pi\sqrt{p^2-i}, \quad . \quad (29)$$

$$i^{-n/2}(\mathrm{ker}_n t + i\,\mathrm{kei}_n t) \doteq p[\mathrm{S}^n + i^n \mathrm{S}^{-n}] \log (\mathrm{S}/i^{1/2})/2\sqrt{p^2-i}. \quad (30)$$

$$\ker t + i\,\mathrm{kei}\, t \doteq p \log (\mathrm{S}/i^{1/2})/\sqrt{p^2-i}, \quad . \quad . \quad . \quad . \quad (31)$$

$$\nu(\mathrm{ker}_\nu t + i\,\mathrm{kei}_\nu t)/t \doteq -\pi i^{-3\nu/2} p[\mathrm{S}^\nu + i^\nu \mathrm{S}^{-\nu}]/2 \sin \nu\pi, \quad . \quad (32)$$

$$n(\mathrm{ker}_n t + i\,\mathrm{kei}_n t)/t \doteq -i^{n/2} p[\mathrm{S}^n - i^n \mathrm{S}^{-n}] \log (\mathrm{S}i^{-1/2})/2, \quad . \quad (33)$$

and

$$(\ker t+i\,\mathrm{kei}\,t)/t \doteqdot -p\{\log(\mathrm{S}i^{-1/2})\}^2/2. \quad . \quad . \quad . \quad . \quad . \quad (34)$$

The well-known operational forms * for $\mathrm{J}_\nu(t)$ and $\mathrm{I}_\nu(t)$ are valid on the contour $c-i\infty$ to $c+i\infty$ provided $\mathrm{R}(\nu)>-1$, but the same forms hold without restriction on ν if the L-contour is that of fig. 1 in the former paper.

3. *Struve, Ster, and Stei Functions* †.

To evaluate the operational form of Struve's function we employ (2). By definition

$$\mathbf{H}_\nu(t)=\sum_{r=0}^{\infty}\frac{(-1)^r(t/2)^{\nu+2r+1}}{\Gamma(r+3/2)\Gamma(\nu+r+3/2)}. \quad . \quad . \quad (35)$$

Applying the formula

$$p\int_0 e^{-pt}t^\nu\,dt=\Gamma(\nu+1)/p^\nu, \quad . \quad . \quad . \quad (36)$$

where $\mathrm{R}(\nu)>-1$, term by term to (35), we find that

$$p\int_0 e^{-pt}\mathbf{H}_\nu(t)\,dt=\sum_{r=0}^{\infty}\frac{(-1)^r\Gamma(\nu+2r+2)}{(2p)^{\nu+2r+1}\Gamma(r+3/2)\Gamma(\nu+r+3/2)}. \quad . \quad . \quad . \quad (37)$$

Writing $\nu=-1$ in (37), we obtain

$$p\int_0^\infty e^{-pt}\mathbf{H}_0(t)\,dt$$

$$=(2p/\pi)\left[p^{-2}-\frac{2}{1.3}p^{-4}+\frac{2.4}{1.3.5}p^{-6}-\ldots\right]. \quad (38)$$

Provided ‡ $|\,p\,|>1$, a condition which can be fulfilled by giving c a suitable value, the series (38) represents § $2p\,\mathrm{cosech}^{-1}\,p/\pi\sqrt{p^2+1}$, so we have

$$\mathbf{H}_0(t)\doteqdot 2p\,\mathrm{cosech}^{-1}\,p/\pi\sqrt{p^2+1}=2p\log(\mathrm{T}/p)/\pi\sqrt{p^2+1}, \quad . \quad . \quad . \quad (39)$$

where $\mathrm{T}=(1+\sqrt{p^2+1})$. We also find that

$$\mathbf{H}_0(at)\doteqdot 2p\,\mathrm{cosech}^{-1}(p/a)/\pi\sqrt{p^2+a^2}$$

$$=2p\log\left(\frac{a+\sqrt{p^2+a^2}}{p}\right)\Big/\pi\sqrt{p^2+a^2}, \quad (40)$$

* Jeffreys, *loc. cit.* (in former paper), pp. 96, 101.

† For the definition of the ster and stei functions, see a paper by the authors, Phil. Mag. xxi. p. 425 (1936).

‡ Referred to the contour $c-i\infty$ to $c+i\infty$.

§ Bromwich, 'Theory of Infinite Series,' 2nd edition (1926). p. 197.

and that

$$\mathbf{H}_1(t) \doteqdot (2/\pi)[1-p^2 \operatorname{cosech}^{-1} p/\sqrt{p^2+1}]$$
$$(=2/\pi)[1-p^2 \log (\mathrm{T}/p)/\sqrt{p^2+1}]. \quad (41)$$

By differentiating the contour integral (65) below [which corresponds to (41)] with respect to t, we obtain

$$\mathbf{H}_1'(t) \doteqdot (2/\pi)[p-p^3 \log (\mathrm{T}/p)/\sqrt{p^2+1}]. \quad . \quad (42)$$

In (41) and (42) the first term on the r.h.s. in brackets cancels that in the expansion of the second. This explains the occurrence of the isolated p in (42),

Writing $\nu=1$ in the recurrence formula

$$\mathbf{H}_{\nu+1}(t)=\mathbf{H}_{\nu-1}(t)-2\mathbf{H}_\nu'(t)+(t/2)^\nu/\sqrt{\pi}\Gamma(\nu+3/2), \quad (43)$$

we get

$$\mathbf{H}_2(t)=\mathbf{H}_0(t)-2\mathbf{H}_1'(t)+2t/3\pi. \quad . \quad . \quad . \quad (44)$$

On replacing the last term in (44) by its operational form, and using (39), (42), we obtain

$$\mathbf{H}_2(t) \doteqdot (2/\pi)[-2p+(1/3p)+(2p^3+p) \log (\mathrm{T}/p)/\sqrt{p^2+1}].$$
$$. \quad . \quad . \quad (45)$$

This procedure can be continued step by step to determine the operational form for any positive integral order, for instance,

$$\mathbf{H}_3(t) \doteqdot (2/\pi)[4p^2+(1/3)+(2/15p^2)$$
$$-(4p^4+3p^2) \log (\mathrm{T}/p)/\sqrt{p^2+1}]. \quad (46)$$

Putting $\nu=1$ in the recurrence formula

$$\nu\mathbf{H}_\nu(t)/t=\mathbf{H}_{\nu-1}(t)-\mathbf{H}_\nu'(t), \quad . \quad . \quad . \quad . \quad (47)$$

and substituting from (39), (42), we find that

$$\mathbf{H}_1(t)/t \doteqdot (2/\pi)[-p+p\sqrt{p^2+1} \log (\mathrm{T}/p)]. \quad . \quad (48)$$

Similarly, we get

$$\mathbf{H}_2(t)/t \doteqdot (2/\pi)[p^2+(1/3)-p^2\sqrt{p^2+1} \log (\mathrm{T}/p)], \quad (49)$$

and

$$\mathbf{H}_3(t)/t \doteqdot (2/\pi)[-(4p^3/3)-(7p/9)+(1/15p)$$
$$+\tfrac{1}{3}p(4p^2+1)\sqrt{p^2+1} \log (\mathrm{T}/p)]. \quad (50)$$

The operational form of ster $t+i$ stei t is obtained from (40) if a is replaced by $i^{3/2}$. Thus

$$\text{ster } t+i \text{ stei } t \doteqdot (2/\pi)p \log\left(\frac{i^{3/2}+\sqrt{p^2-i}}{p}\right)\Big/\sqrt{p^2-i}. \quad (51)$$

Operational forms for the modified Struve function

$$\mathbf{L}_\nu(t)=i^{-(\nu+1)}\mathbf{H}_\nu(ti)=\sum_{r=0}^{\infty}\frac{(t/2)^{\nu+2r+1}}{\Gamma(r+3/2)\Gamma(\nu+r+3/2)}, \quad (52)$$

can be derived independently or deduced from the preceding forms with the following results :

$$\mathbf{L}_0(t)\doteqdot 2p \operatorname{cosec}^{-1} p/\pi\sqrt{p^2-1}=2p \sin^{-1}(1/p)/\pi\sqrt{p^2-1}, \quad . \quad . \quad . \quad (53)$$

$$\mathbf{L}_1(t)\doteqdot(2/\pi)[-1+p^2 \operatorname{cosec}^{-1} p/\sqrt{p^2-1}], \quad . \quad . \quad . \quad . \quad (54)$$

$$\mathbf{L}_2(t)\doteqdot(2/\pi)[-2p-(1/3p)+(2p^3-p) \operatorname{cosec}^{-1} p/\sqrt{p^2-1}], \quad . \quad . \quad . \quad (55)$$

$$\mathbf{L}_3(t)\doteqdot(2/\pi)[-4p^2+(1/3)-(2/15p^2)+(4p^4-3p^2) \operatorname{cosec}^{-1} p/\sqrt{p^2-1}], \quad (56)$$

$$\mathbf{L}_1(t)/t\doteqdot(2/\pi)[p-p\sqrt{p^2-1} \operatorname{cosec}^{-1} p], \quad . \quad . \quad . \quad . \quad . \quad (57)$$

$$\mathbf{L}_2(t)/t\doteqdot(2/\pi)[p^2-(1/3)-p^2\sqrt{p^2-1} \operatorname{cosec}^{-1} p], \quad . \quad . \quad (58)$$

and

$$\mathbf{L}_3(t)/t\doteqdot(2/\pi)[(4p^3/3)-(7p/9)-(1/15p)-\tfrac{1}{8}p(4p^2-1)\sqrt{p^2-1} \operatorname{cosec}^{-1} p]. \quad (59)$$

In this section all operational forms satisfy the condition $\lim_{|p|\to\infty} |\phi(p)/p| \to 0$, so the integrals in § 4 are valid.

4. *Contour Integrals for Struve Functions.*

From (1) and (39), we have

$$\mathbf{H}_0(t)=\frac{(2/\pi)}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{zt}\log\left(\frac{1+\sqrt{z^2+1}}{z}\right)dz/\sqrt{z^2+1}. \quad (60)$$

Putting $z=(\zeta-1/\zeta)/2$ and writing z for ζ thereafter, we have for t real and positive

$$\mathbf{H}_0(t)=\frac{(2/\pi)}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{t(z-1/z)/2}\log[(z+1)/(z-1)]\,dz/z \quad (61)$$

$$=\frac{(2/\pi)}{2\pi i}\int_{c\ \ i\infty}^{c+i\infty} e^{z-t^2/4z}\log[(2z+t)/(2z-t)]\,dz/z, \quad (62)$$

when z is written for $tz/2$ in (61).

From (40), we have

$$\mathbf{H}_0(at)=\frac{(2/\pi)}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{zt}\log\left(\frac{a+\sqrt{z^2+a^2}}{z}\right)dz/\sqrt{z^2+a^2}, \quad (63)$$

and by substituting $a=i^{3/2}$, or from (51), we get

$$\mathbf{H}_0(ti^{3/2}) = \text{ster}\, t + i\,\text{stei}\, t$$

$$= \frac{(2/\pi)}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{zt}\log\left(\frac{i^{3/2}+\sqrt{z^2-i}}{z}\right)dz/\sqrt{z^2-i}. \quad (64)$$

From (41), (45), (46) we obtain the following integral representations :

$$\mathbf{H}_1(t)=\frac{(2/\pi)}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{zt} \times\left[(1/z)-z\log\left(\frac{1+\sqrt{z^2+1}}{z}\right)\Big/\sqrt{z^2+1}\right]dz, \quad (65)$$

$$\mathbf{H}_2(t)=\frac{(2/\pi)}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{zt}\left[-2+\frac{1}{3z^2} +\frac{(2z^2+1)}{\sqrt{z^2+1}}\log\left(\frac{1+\sqrt{z^2+1}}{z}\right)\right]dz, \quad (66)$$

and

$$\mathbf{H}_3(t)=\frac{(2/\pi)}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{zt}\left[4z+\frac{1}{3z}+\frac{2}{15z^3} -\frac{(4z^3+3z)}{\sqrt{z^2+1}}\log\left(\frac{1+\sqrt{z^2+1}}{z}\right)\right]dz. \quad (67)$$

In (66) and (67) the terms -2 and $4z$ cancel with corresponding terms of opposite sign in the expansion of the logarithmic element. So long as operational forms are not in question these terms can be omitted from the integrands.

By aid of the operational forms already deduced, contour integrals for

$$\mathbf{H}_1(t)/t \ldots \mathbf{H}_3(t)/t,\ \mathbf{L}_0(t) \ldots \mathbf{L}_3(t),\ \mathbf{L}_1(t)/t \ldots \mathbf{L}_3(t)/t$$

can be written down immediately.

5. *Integral Bessel Functions.*

We define the integral function corresponding to a function $f(t)$ as

$$fi(t)=\int_t^\infty f(t)\,dt/t \quad . \quad . \quad . \quad . \quad . \quad (68)$$

Van der Pol * has shown that

$$\int_t^\infty f(t)\,dt/t \doteqdot \int_0^p \phi(p)\,dp/p, \quad . \quad . \quad . \quad (69)$$

where $\phi(p)$ is the operational form of $f(t)$ obtained from (2) and, therefore, referred to the contour $c-i\infty$ to $c+i\infty$. Lowry † has shown that this relationship is valid for the L′ contour, and it can be shown to hold for the L-contour also ‡. Applying (69) we have

$$\mathrm{Y}i_\nu(t) \doteqdot \frac{1}{\sin \nu\pi}\int_0^p [\cos \nu\pi(\sqrt{p^2+1}-p)^\nu - (\sqrt{p^2+1}+p)^\nu]\,dp/\sqrt{p^2+1}. \quad (70)$$

Substituting $p=\sinh u$ in (70) we find that

$$\mathrm{Y}i_\nu(t) \doteqdot [(1+\cos \nu\pi)-(\sqrt{p^2+1}+p)^\nu - (\sqrt{p^2+1}-p)^\nu \cos \nu\pi]/\nu \sin \nu\pi. \quad (71)$$

Proceeding to the limit $\nu \to n$,

$$\mathrm{Y}i_n(t) \doteqdot [(\sqrt{p^2+1}-p)^n + (-1)^{n+1}(\sqrt{p^2+1}+p)^n]\log(\sqrt{p^2+1}+p)/n\pi, \quad (72)$$

and in particular

$$\mathrm{Y}i_0(t) \doteqdot -\{\log(p+\sqrt{p^2+1})\}^2/\pi = -(\sinh^{-1} p)^2/\pi. \quad (73)$$

In like manner we also deduce that

$$\mathrm{I}i_\nu(t) \doteqdot -[(p-\sqrt{p^2-1})^\nu - e^{-\nu\pi i/2}]/\nu. \quad . \quad . \quad . \quad . \quad . \quad (74)$$

$$\mathrm{I}i_n(t) \doteqdot (p-\sqrt{p^2-1})^n \log(p+\sqrt{p^2-1}) - i^{-n}\log i, \quad (75)$$

$$\mathrm{I}i_0(t) \doteqdot -\log i(p-\sqrt{p^2-1}) = \cosh^{-1} p - \tfrac{1}{2}\pi i, \quad . \quad . \quad (76)$$

$$\mathrm{K}i_\nu(t) \doteqdot \pi[(p+\sqrt{p^2-1})^\nu + (p+\sqrt{p^2-1})^{-\nu} - 2\cos(\nu\pi/2)]/2\nu \sin \nu\pi, \quad (77)$$

$$\mathrm{K}i_n(t) \doteqdot (-1)_n\{[(p+\sqrt{p^2-1})^n - (p+\sqrt{p^2-1})^{-n}]\log(p+\sqrt{p^2-1}) + \pi \sin(n\pi/2)\}/2n, \quad . \quad . \quad . \quad (78)$$

and

$$\mathrm{K}i_0(t) \doteqdot [\{\log(p+\sqrt{p^2-1})\}^2 - \{\log i\}^2]/2 = [(\cosh^{-1} p)^2/2] + \tfrac{1}{8}\pi^2. \quad (79)$$

* Phil. Mag. xiii. p. 537 (1932).
† *Loc. cit.*
‡ See Phil. Mag. xxiii. p. 762 (1937).

XCI. *Operational Representation of the Parabolic Cylinder Functions.*—Second Paper. *By* R. S. VARMA *.

1. IN a recent issue [1] of this Journal I have obtained some new properties of Weber's parabolic cylinder function by means of its operational representation. The object of this paper is to give some new forms of operational images for the parabolic cylinder function and to evaluate some infinite integrals involving the function.

2. We know [2] that, when n is a positive integer,

$$D_n^2(x) = \frac{1}{2^n} \sum_{r=0}^{n} \frac{n!\,1.3.5\ldots(2r-1)}{r!\,(n-r)!} D_{2n-2r}(x\sqrt{2}),$$

which can be written as

$$x^{-\frac{1}{2}}e^{\frac{1}{2}x}D_n^2(\sqrt{x}) = \frac{1}{2^n} \sum_{r=0}^{n} \frac{n!\,(2r)!}{2^r(r!)^2(n-r)!} x^{-\frac{1}{2}}e^{\frac{1}{2}x}D_{2n-2r}(\sqrt{2x}).$$

Using the known representation [3]

$$x^{-\frac{1}{2}}e^{\frac{1}{2}x}D_{2n}(\sqrt{2x}) \doteqdot \sqrt{\pi}\,\frac{(2n)!}{2^n n!}\,p^{\frac{1}{2}}\left(\frac{1}{p}-1\right)^n,$$

we obtain the following image :

$$x^{-\frac{1}{2}}e^{\frac{1}{2}x}D_n^2(\sqrt{x})$$
$$\doteqdot \frac{\sqrt{\pi}}{2^{2n}} \sum_{r=0}^{n} \frac{n!\,(2r)!\,(2n-2r)!}{(r!)^2(n-r)!} \sum_{s=0}^{n-r} \frac{(-)^s}{s!\,(n-r-s)!} \frac{1}{p^{n-r-s-\frac{1}{2}}}.$$

3. Elsewhere [4] I have shown that

$$\Gamma(m)pe^{\frac{1}{2}p^2}D_{-m}(p) \doteqdot x^{m-1}e^{-\frac{1}{2}x^2}, \quad [R(m)>0],$$

which can be written in the form

$$\Gamma(m+1)pe^{\frac{1}{2}p^2}D_{-m-1}(p) \doteqdot x^m e^{-\frac{1}{2}x^2}, \quad [R(m)>-1]. \quad (3.1)$$

Now Carson [5] has shown that, if

$$\phi(p) \doteqdot f(x),$$

then

$$p\phi(p) \doteqdot \frac{d}{dx}f(x), \quad \text{if } f(o)=0.$$

* Communicated by the Author.

Repeated application of this theorem to (3.1) gives

$$\Gamma(m+1)p^{\lambda+1}e^{\frac{1}{4}p^2}\mathrm{D}_{-m-1}(p)$$

$$\doteqdot e^{-\frac{1}{4}x^2}\sum_{r=0}^{\lambda}(-)^{\lambda-r}{}_{\lambda}\mathrm{C}_r m(m-1)\ldots(m-r+1)x^{m-r}\mathrm{D}_{\lambda-r}(x),$$

where we have used the known result [6]

$$\mathrm{D}_n(x)=(-)^n e^{\frac{1}{4}x^2}\frac{d^n}{dx^n}(e^{-\frac{1}{2}x^2}).$$

In particular when $\lambda=m$ we get

$$\Gamma(m+1)p^{m+1}e^{\frac{1}{4}p^2}\mathrm{D}_{-m-1}(p)$$

$$\doteqdot e^{-\frac{1}{4}x^2}\sum_{r=0}^{m}(-)^{m-r}\frac{(m!)^2}{r!\,\{(m-r)!\}^2}x^{m-r}\mathrm{D}_{m-r}(x).$$

4. A form of Parseval's theorem recently given by Goldstein [3] is that, if $\phi_1(p)$ and $\phi_2(p)$ are the operational images of $f_1(x)$ and $f_2(x)$ respectively, then

$$\int_0^\infty \frac{\phi_1(x)f_2(x)}{x}\,dx=\int_0^\infty \frac{\phi_2(x)f_1(x)}{x}\,dx, \quad . \quad . \quad (4.1)$$

provided the integrals converge.

In the above theorem take

$$\phi_1(p)=\Gamma(m)pe^{\frac{1}{4}p^2}\mathrm{D}_{-m}(p),$$

$$f_1(x)=x^{m-1}e^{-\frac{1}{4}x^2}; \qquad \mathrm{R}(m)>0,$$

$$\phi_2(p)=\frac{1}{p^n},$$

$$f_2(x)=\frac{x^n}{\Gamma(n+1)}; \qquad \mathrm{R}(n)>-1,$$

we then obtain

$$\int_0^\infty x^n e^{\frac{1}{4}x^2}\mathrm{D}_{-m}(x)\,dx=\frac{\Gamma(n+1)}{\Gamma(m)}\int_0^\infty x^{m-n-2}e^{-\frac{1}{4}x^2}dx.$$

Since

$$\int_0^\infty e^{-x}x^{s-1}\,dx=\Gamma(s), \qquad \mathrm{R}(s)>0,$$

we deduce that

$$\int_0^\infty x^n e^{\frac{1}{4}x^2}\mathrm{D}_{-m}(x)\,dx=\frac{\Gamma(n+1)\Gamma(\frac{1}{2}m-\frac{1}{2}n-\frac{1}{2})}{2^{-\frac{1}{2}m+\frac{1}{2}n+\frac{3}{2}}\Gamma(m)},$$

$$\mathrm{R}(n)>-1, \text{ and } \mathrm{R}(m-n-1)>0,$$

a result obtained otherwise by me [7] recently.

5. Now in (4.1) take

$$\phi_1(p) = \frac{\sqrt{\pi}}{2^{2n}} \sum_{r=0}^{n} \frac{n!\,(2r)!\,(2n-2r)!}{(r!)^2\,(n-r)!} \sum_{s=0}^{n-r} \frac{(-)^s}{s!\,(n-r-s)!} \frac{1}{p^{n-r-s-\frac{1}{2}}},$$

$$f_1(x) = x^{-\frac{1}{2}} e^{\frac{1}{2}x} \mathrm{D}_n^2(\sqrt{x});$$

$$\phi_2(p) = \Gamma(m) p e^{\frac{1}{2}p^2} \mathrm{D}_{-m}(p),$$

$$f_2(x) = x^{m-1} e^{-\frac{1}{2}x^2};$$

we then obtain

$$\int_0^\infty t^{-\frac{1}{2}} e^{\frac{1}{2}t + \frac{1}{4}t^2} \mathrm{D}_n^2(\sqrt{t}) \mathrm{D}_{-m}(t)\, dt$$

$$= \frac{\sqrt{\pi}}{2^{2n}\Gamma(m)} \sum_{r=0}^{n} \frac{n!\,(2r)!\,(2n-2r)!}{(r!)^2\,(n-r)!} \sum_{s=0}^{n-r} \frac{(-)^s}{s!\,(n-r-s)!} \times \int_0^\infty x^{m-n+r+s-\frac{3}{2}} e^{-\frac{1}{4}x^2} dx$$

$$= \frac{\sqrt{\pi}}{2^{2n}\Gamma(m)} \sum_{r=0}^{n} \frac{n!\,(2r)!\,(2n-2r)!}{(r!)^2\,(n-r)!} \sum_{s=0}^{n-r} \frac{(-)^s}{s!\,(n-r-s)!} \times 2^{\frac{1}{2}(m-n+r+s-\frac{1}{2})} \Gamma(\tfrac{1}{2}m - \tfrac{1}{2}n + \tfrac{1}{2}r + \tfrac{1}{2}s - \tfrac{1}{4}),$$

true when

$$\mathrm{R}(m-n) > \tfrac{1}{2}, \quad n \text{ being integral.}$$

References.

(1) R. S. Varma, "Operational Representation of the Parabolic Cylinder Function," Phil. Mag. xxii. pp. 29–34 (1936).

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XCII. *On some Effects caused in the Ionosphere by Electric Waves.*—Part I.

By V. A. Bailey, *M.A., D.Phil (Oxon), F.Inst.P., Professor of Experimental Physics, University of Sydney* †.

1. *Introduction.*

In a recent communication to this Journal ‡, a general theory has been given of the action of variable electric fields on electrons in a gas in the presence of a constant magnetic field.

One of the results of this investigation is that an error is revealed in the theory, previously advanced §, of the influence of an electric wave on the ionosphere, so in the present communication the latter problem is considered afresh on a more rigorous foundation.

It will be seen that the present theory is in better agreement with observation than the old, and that it also leads to the prediction of a hitherto unknown phenomenon.

2. *Mean Work $\overline{w}$ done on an Electron Per Collision by an Elliptic Wave in the Ionosphere.*

Since in general a wave traversing the ionosphere is elliptically polarized, we need to derive in a form convenient for our purpose the essential characteristics of such a wave; the notation adopted is given in the following table :—

Notation.

ω=angular frequency of the wave.

$\breve{E}$, $\breve{H}$=electric and magnetic vectors, in e.s.u. and e.m.u. respectively.

$\breve{J}$=vector of electronic current density in e.m.u.

$\breve{u}$=mean velocity vector of a group of electrons.

ν=average number of collisions per second made by an electron.

† Communicated by the Author.

‡ V. A. Bailey, "The Motions of Electrons in a Gas in the Presence of Variable Electric Fields and a Constant Magnetic Field," Phil. Mag. xxiii. p. 774 (April 1937).

§ V. A. Bailey and D. F. Martyn, Phil. Mag. vol. xviii. p. 369 (Aug. 1934),

$\breve{A}$, V =vector and scalar potentials †.

$\breve{\nabla} = \breve{i}\frac{\partial}{\partial x} + \breve{j}\frac{\partial}{\partial y} + \breve{k}\frac{\partial}{\partial z}$, where $\breve{i}$, $\breve{j}$, $\breve{k}$ are orthogonal unit vectors, parallel respectively to the x, y, and z axes.

$\Box^2 = \nabla^2 - \frac{1}{c^2}D^2$, $\quad D = \frac{\partial}{\partial t}$, $\quad \partial = D + \nu$.

e, m =electronic charge and mass respectively, in e.s.u.

$\sigma = -e/m$.

$\tau = e^2/m = 2{\cdot}54 \times 10^8$.

Ω=He/m, where H is the total intensity of the earth's magnetic field.

$\breve{\Omega} = \Omega\breve{h}$, where $\breve{h}$ is a unit vector in the direction of the earth's magnetic field.

N=number of electrons per c.c.

$p_0^2 = 4\pi N\tau = 3{\cdot}19 \times 10^9 N$.

μ=index of refraction.

κ=coefficient of absorption.

$cq = \mu - \iota\chi$, where $\chi = \kappa c/\omega$.

$\zeta = c^2q^2 - 1 = \xi + \iota\eta$, where ξ and η are real.

$\omega_0 = p_0^2/\omega$.

Z=root mean square value of $\breve{E} = E/\sqrt{2}$.

θ=angle between the wave's direction of propagation and the earth's magnetic field vector.

We will determine the form of such an elliptic wave in a homogeneous ionosphere by means of the vector and scalar potentials $\breve{A}$ and V, and the following relations:

$$DV = -c\breve{\nabla}\breve{A}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\breve{E} = -c^{-1}D\breve{A} - \breve{\nabla}V, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\breve{H} = V\breve{\nabla}\breve{A}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$\breve{J} = -(4\pi)^{-1}\Box^2\breve{A}, \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$\breve{J} = (Ne/c)\breve{u}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and, as in equation (1) of the previous communication ‡,

$$\partial\breve{u} = -\sigma\breve{E} + V\breve{\Omega}\breve{u}. \quad . \quad . \quad . \quad . \quad . \quad (6)$$

† V before two vectors is also used to indicate a vector product.

‡ V. A. Bailey, *loc. cit.*

Adopting a usual procedure we now choose the x-axis to lie in the direction of propagation of the wave, and therefore set

$$\breve{\mathrm{A}}=\breve{\mathscr{A}}e^{i\omega(t-qx)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

and in this and other expressions for vectors take only the real part.

We may therefore in all the above equations make the following substitution :

$$\mathrm{D}=\iota\omega, \quad \breve{\nabla}=-\iota\omega q\breve{i}, \quad \Box^2=-\omega^2\zeta/c^2.$$

Also $\breve{\mathrm{A}}=\breve{\mathrm{A}}_\mathrm{L}+\breve{\mathrm{A}}_\mathrm{T}$ where $\breve{\mathrm{A}}_\mathrm{L}$ is the component in the direction of x and $\breve{\mathrm{A}}_\mathrm{T}$ is the component transverse to x.

Therefore

$$\mathrm{V}=cq\mathrm{A}_\mathrm{L}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

$$\breve{\mathrm{E}}=-(\iota\omega/c)(\breve{\mathrm{A}}_\mathrm{T}-\zeta\breve{\mathrm{A}}_\mathrm{L}), \quad . \quad . \quad . \quad . \quad (9)$$

$$\breve{\mathrm{H}}=-\iota\omega q\mathrm{V}\breve{i}\breve{\mathrm{A}}_\mathrm{T}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

$$\breve{\mathrm{J}}=(\omega^2/4\pi c^2)\zeta\breve{\mathrm{A}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

$$\breve{u}=-(\omega\sigma/c\omega_0)\zeta\breve{\mathrm{A}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

On eliminating $\breve{u}$ and E between (6), (12), and (9) we obtain the following equation in $\breve{\mathscr{A}}$:

$$a\breve{\mathscr{A}}+b\breve{\mathscr{A}}_\mathrm{L}-\mathrm{V}\breve{\Omega}\breve{\mathscr{A}}=0, \quad . \quad . \quad . \quad . \quad (13)$$

where

$$\left.\begin{aligned} a&=\nu+\iota\omega+\iota\omega_0\zeta^{-1},\\ b&=-\iota\omega_0(1+\zeta^{-1}).\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (14)$$

To solve (13) we begin by noticing that it is identical with equation (1) of the previous communication† when in the latter we set

$$\breve{u}=\breve{\mathscr{A}}, \quad \partial=a, \quad \text{and} \quad \sigma\breve{\mathrm{E}}=b\breve{\mathscr{A}}_\mathrm{L}.$$

So, on making these substitutions in equation (2) of that communication, we obtain the following solution of (13) :

$$-a(a^2+\Omega^2)\breve{\mathscr{A}}=b\mathscr{A}_\mathrm{L}\{a^2\breve{i}+a\Omega\mathrm{V}\breve{h}\breve{i}+\Omega^2\breve{h}(\breve{h}\breve{i})\}. \quad (15)$$

† V. A. Bailey, *loc. cit.*

We now select the axes of coordinates, so that $\breve{h}$ lies in the xz-plane and makes an angle θ with $\breve{i}$.

Then (15) becomes

$$-a(a^2+\Omega^2)\cdot\breve{\mathscr{A}}=b\mathscr{A}_{\mathrm{L}}\{a^2\breve{i}+a\Omega_{\mathrm{T}}\breve{j}+\Omega_{\mathrm{T}}(\Omega_{\mathrm{L}}\breve{i}+\Omega_{\mathrm{T}}\breve{k}),$$

where $$\breve{\Omega}=\Omega_{\mathrm{L}}\breve{i}+\Omega_{\mathrm{T}}\breve{k}.$$

This is equivalent to the following three relations :

$$-a(a^2+\Omega^2)=b(a^2+\Omega_{\mathrm{L}}^2), \quad . \quad . \quad . \quad (16)$$

$$-a(a^2+\Omega^2)\mathscr{A}_y=ba\Omega_{\mathrm{T}}\mathscr{A}_x, \quad . \quad . \quad . \quad (17)$$

$$-a(a^2+\Omega^2)\mathscr{A}_z=b\Omega_{\mathrm{L}}\Omega_{\mathrm{T}}\mathscr{A}_x. \quad . \quad . \quad . \quad (18)$$

Equation (16) may be expressed in the form

$$(a+b)(a^2+\Omega_{\mathrm{L}}^2)+a\Omega_{\mathrm{T}}^2=0,$$

and so, by means of (14), in the form

$$\left.\begin{aligned}\mathrm{R}+\mathrm{R}^{-1}&=\frac{\Omega_{\mathrm{T}}^2/\Omega_{\mathrm{L}}}{\nu+i(\omega-\omega_0)},\\ \text{where}\qquad \mathrm{R}&=-a/\Omega_{\mathrm{L}}.\end{aligned}\right\} \quad . \quad . \quad . \quad (19)$$

From (17) and (18) we have

$$\frac{\mathscr{A}_x}{(\mathrm{R}^2+1)\cot\theta}=\frac{\mathscr{A}_y}{-\mathrm{R}}=\mathscr{A}_z,$$

and so from (10) it follows that

$$\mathrm{R}=\mathrm{H}_z/\mathrm{H}_y. \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

Thus (19) yields the number R, which specifies the polarization of the wave under consideration.

This is a well-known result.

The value of $\overline{w}$, the mean work done per collision, can now be determined by substituting in (21) of the previous communication † the real part of the expression for $\breve{\mathrm{E}}$ given here in (9), but the following alternative procedure is more convenient.

It is easy to verify that if

$$f=\mathrm{F}e^{i\omega t} \quad \text{and} \quad g=\mathrm{G}e^{i\omega t},$$

† V. A. Bailey, *loc. cit.*

then the mean value of the product $\mathscr{R}(f)\mathscr{R}(g)$ of the real parts is given by

$$\tfrac{1}{2}\mathscr{R}(fg^*) \quad \text{or} \quad \tfrac{1}{2}\mathscr{R}(f^*g),$$

where the asterisk (*) appended to a symbol indicates that its conjugate is to be taken.

Since, by the definition of $\overline{w}$, $w\nu/\tau$ is equal to the mean value of $\breve{\mathrm{E}}\breve{u}/\sigma$, it therefore follows that

$$\overline{w}\nu/\tau = \tfrac{1}{2}\mathscr{R}(\mathrm{E}\breve{u}^*/\sigma),$$

and so

$$\begin{aligned}\overline{w}\left(\frac{2\nu\omega_0 c^2}{\tau\omega^2}\right) &= \mathscr{R}\{\iota\zeta^*(\breve{\mathrm{A}}_\mathrm{T}\breve{\mathrm{A}}_\mathrm{T}^*) - \iota(\zeta\zeta^*)(\breve{\mathrm{A}}_\mathrm{L}\breve{\mathrm{A}}_\mathrm{L}^*)\} \\ &= -\eta(\breve{\mathrm{A}}_\mathrm{T}\breve{\mathrm{A}}_\mathrm{T}^*) \\ &= -\eta\,|\,\mathscr{A}_\mathrm{T}\,|^2 e^{-2\kappa x}, \quad . \quad . \quad . \quad . \quad . \quad (21)\end{aligned}$$

where $\xi+\iota\eta=\zeta$, *i. e.*, where $-\eta=2\mu\chi$.

This formula can be put in a more useful form as follows: Let U denote the amount of radiant energy which passes per second through one square centimetre of a surface which is normal to the x-axis, and let $\breve{\mathrm{P}}$ denote Poynting's vector.

Therefore

$$\mathrm{U}=\breve{\imath}\breve{\mathrm{P}} \quad \text{and} \quad \breve{\mathrm{P}}=\frac{c}{4\pi}\mathrm{V}\breve{\mathrm{E}}\breve{\mathrm{H}}.$$

On taking mean values we then have

$$\begin{aligned}\overline{\breve{\mathrm{P}}} &= \frac{c}{8\pi}\mathscr{R}(\mathrm{V}\breve{\mathrm{E}}\breve{\mathrm{H}}) \\ &= \frac{c}{8\pi}\mathscr{R}\left\{\frac{\omega^2}{c}q^*\mathrm{V}(\breve{\mathrm{A}}_\mathrm{T}-\zeta\breve{\mathrm{A}}_\mathrm{L})(\mathrm{V}\breve{\imath}\breve{\mathrm{A}}_\mathrm{T}^*)\right\} \\ &= \frac{\omega^2}{8\pi c}\mathscr{R}[cq^*\{\breve{\imath}(\breve{\mathrm{A}}_\mathrm{T}\breve{\mathrm{A}}_\mathrm{T})+\breve{\mathrm{A}}_\mathrm{T}^*\zeta(\breve{\mathrm{A}}_\mathrm{L}\breve{\imath})\}],\end{aligned}$$

and so

$$\overline{\mathrm{U}}=\frac{\omega^2\mu}{8\pi c}(\breve{\mathrm{A}}_\mathrm{T}\breve{\mathrm{A}}_\mathrm{T}^*), \quad . \quad . \quad . \quad . \quad . \quad (22)$$

i. e.,

$$\overline{\mathrm{U}}=\frac{\omega^2\mu}{8\pi c}\,|\,\mathscr{A}\,|^2 e^{-2\kappa x}. \quad . \quad . \quad . \quad . \quad (23)$$

Combining this result with (21) we therefore have

$$\bar{w} = \frac{8\pi\tau\chi}{cv\omega} \cdot \bar{U}, \quad \ldots \ldots \quad (24)$$

or

$$\bar{w} = \frac{2\kappa}{N\nu} \cdot \bar{U}. \quad \ldots \ldots \quad (25)$$

This formula for $\bar{w}$ may also be derived independently of (21) by means of Poynting's Theorem, as is shown in Appendix I.

Finally, we will derive another expression for U, which will be needed later.

From equation (10) we have the following results :

$$H_x = 0, \quad H_y = \iota\omega q A_z, \quad H_z = -\iota\omega q A_y.$$

Hence

$$\breve{A}_T \breve{A}_T^* = A_y A_y^* + A_z A_z^*$$

$$= \frac{1}{\omega^2 q q^*} (H_y H_y^* + H_z H_z^*)$$

$$= \frac{c^2}{\omega^2(\mu^2+\chi^2)} (|H_y|^2 + |H_z|^2),$$

and so, with the help of (21), the relation (22) may be put in the form

$$\bar{U} = \frac{c}{8\pi} \cdot \frac{\mu}{\mu^2+\chi^2} \cdot |H|^2 (1+|R|^2). \quad \ldots \quad (26)$$

3. *The Variation of $\bar{w}$ with the Wave-length.*

A complete discussion of the variation of $\bar{w}$ with the different factors involved in the expression $2\kappa/N\nu$ would be too lengthy to give here, so we limit the discussion in the present communication to that of some of the simpler and more interesting consequences of the formula (25).

A preliminary notion of the variation of $\bar{w}$ with the wave-length may be obtained without much trouble by using the curves of dispersion and absorption which have been published by Dr. Martyn †. Thus for an extraordinary wave of mean energy flux density $\bar{U}=1$ erg/sec./cm.^{2} travelling in a direction inclined at an angle $\theta=45°$ to

† D. F. Martyn, Phil. Mag., February 1935, p. 376.

a magnetic field whose gyro-frequency $\Omega/2\pi$ corresponds to a wave-length of 200 metres, and in a medium containing $N=10^4$ electrons per cm.^{3}, we have the results shown in Table I. The values of κ are obtained from Martyn's curves corresponding to the two values of ν: 10^5 and 10^6, and the values of $\bar{w}$ are then calculated by means of (25).

This table shows that $\bar{w}$ is least for the lowest wave-length and increases steadily with the wave-length except in the neighbourhood of the gyro wave-length λ_H; at this wave-length a striking maximum value is attained by w, *i. e.*, a notable degree of resonance appears†.

TABLE I.

Extraordinary Wave.

$\bar{U}=1.\quad \theta=45°.\quad \lambda_H=200.\quad N=10^4,$

λ. metres.	$\nu=10^5$.		$\nu=10^6$.	
	$\kappa\times10^6$.	$\bar{w}\times10^{15}$.	$\kappa\times10^6$.	$\bar{w}\times10^{15}$.
100	1	2	3	0·6
200	2000	4000	300	60
400	2	4	10	2
2000	30	60	30	6

The corresponding numbers for an ordinary wave are shown in Table II. From this it will be seen that for the ordinary wave w also increases with the wave-length, but that there is no indication of any resonance.

These conclusions are supported by treating the problem of determining $\bar{w}$ for an ionospheric wave as an example of the general problem solved in the previous communication ‡ and considering the appropriate case§ of the formula (21) for w given there, namely:

$$\bar{w} = \tau Z^2\left[\frac{\frac{1}{2}s^2}{\nu^2+(\omega-\Omega)^2}+\frac{\frac{1}{2}s^2}{\nu^2+(\omega+\Omega)^2}+\frac{c^2}{\nu^2+\omega^2}\right], \quad (27)$$

† That for the extraordinary wave a notable maximum in κ occurs near the gyro-frequency was concluded as early as April 1925 by Appleton and Barnett, and by Nicholls and Schelleng.

‡ V. A. Bailey, *loc. cit.* See also the formula (21.1).

§ That corresponding to a plane wave.

where $s=\sin\phi$, $c=\cos\phi$, and ϕ is the angle between the electric vector and the earth's local magnetic vector.

If we here neglect the complicated effects on s and c, which may be produced as the wave-length changes, this formula (27) leads by inspection to the following conclusions :—

(a) When $\nu<<\Omega$ and $\omega<<\Omega$, and when the constituent electric vector makes angles less than 90° with the earth's magnetic vector, the dominant term in the expression for w is proportional to $1/(\nu^2+\omega^2)$, *i. e.*, for wave-lengths exceeding that corresponding to the gyro-frequency $\bar{w}$ usually increases with the wave-length.

TABLE II.

Ordinary Wave.

$\bar{U}=1$. $\theta=45°$. $\lambda_H=200$. $N=10^4$.

λ. metres.	$\nu=10^5$.		$\nu=10^6$.	
	$\kappa\times10^6$.	$w\times10^{15}$.	$\kappa\times10^6$.	$w\times10^{15}$.
100	<1	<2	1	0·2
200	0·7	1·4	4	0·8
400	5	10	25	5
2000	70	140	70	14

(b) When $\nu<<\Omega$ and ω passes through the value Ω then w usually passes through a prominent maximum value, *i. e.*, a notable degree of resonance occurs.

(c) When $\omega>>\Omega$ the value of w decreases rapidly with the wave-length and is much less than in (a).

4. *The Modulation impressed by a Long-Wave Broadcasting Radio Station on the Wave from another Station* †.

For the long-wave station we have in general

$$\nu<<\Omega, \quad \omega<<\Omega. \quad . \quad . \quad . \quad . \quad . \quad (28)$$

Also we may assume, as a first approximation, that the electric field in the part of the E-layer considered is the same as the field produced by the emitted plane wave at places just below the E-layer. We may therefore

† For some of the data and formulæ used in this section reference should be made to V. A. Bailey and D. F. Martyn, Phil. Mag., August 1934, p. 369.

adopt (27) as the formula for $\bar{w}$ corresponding to such a wave.

For most parts of the E-layer which are notably subjected to the emission from a typical aerial system, the values of s^2 and c^2 are of the same order; we may therefore, without serious error, assign to each quantity a value equal to the mean value $\frac{1}{2}$.

On taking (28) into account we thus obtain the following approximate formula:

$$\bar{w} = \frac{\frac{1}{2}\tau Z^2}{\nu^2+\omega^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

This formula is independent of the earth's magnetic field. It may, therefore, be usefully compared with the result of substituting in (25) the well known formula for κ derived from Lorentz's theory of dispersion, namely,

$$\kappa = \frac{\nu}{2c\mu} \cdot \frac{4\pi N e^2}{m(\nu^2+\omega^2)},$$

and setting $\mu \doteq 1$. The resulting formula is identical with (29).

To determine the temporal variation $\Delta\nu$ of the collision frequency ν we may now proceed as follows.

The equations which describe the relation of ν to $\bar{w}$ are those numbered (24) and (25) in the previous communication †, namely,

$$a\mathrm{D}\nu+\eta=\bar{w} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

and

$$\eta_0=w_0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

where $a=ml^2$, l is the mean free path of an electron, η is the energy lost by an electron at a collision, and the subscript zero attached to a symbol indicates the value of the corresponding quantity in the steady state.

If the temporal variation $\Delta\bar{w}$ of $\bar{w}$ about w_0 be small we may then write

$$\left.\begin{aligned} \bar{w} &= w_0+\Delta\bar{w}, \\ \nu &= \nu_0+\Delta\nu, \\ \eta &= \eta_0+a\mathrm{R}.\Delta\nu, \\ \mathrm{R} &= a^{-1}\frac{d\eta_0}{d\nu_0}, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (32)$$

where $\Delta\nu$ is also small.

† V. A. Bailey, *loc. cit.* For simplicity we here omit the bars over ν and η.

On subtracting (31) from (30) we obtain the following differential equation for determining $\Delta\nu$:

$$\frac{d}{dt}\Delta\nu+\mathrm{R}.\Delta\nu=a^{-1}\Delta\bar{w}. \quad . \quad . \quad . \quad (33)$$

If the wave is modulated at a musical frequency $n/2\pi$ as follows :

$$\mathrm{Z}=\mathrm{Z}_0(1+\mathrm{M}\sin nt), \quad \text{where } \mathrm{M}<1,$$

we have

$$\Delta\bar{w}=2\mathrm{M}w_0\sin nt-\tfrac{1}{2}\mathrm{M}^2w_0\cos 2nt.$$

It then follows from (33) that the permanent part of $\Delta\nu$ is given approximately by the formula

$$\Delta\nu=\frac{2\mathrm{M}w_0}{a\sqrt{\mathrm{R}^2+n^2}}\sin(nt-\phi), \quad . \quad . \quad . \quad (34)$$

where $$\tan\phi=n/\mathrm{R}.$$

Thus ν is modulated at the frequency $n/2\pi$ and with a coefficient of modulation M′ where

$$\mathrm{M}'=2\mathrm{M}w_0\delta/a\mathrm{R}\nu_0 \quad . \quad . \quad . \quad . \quad . \quad (35)$$

and

$$\delta=\frac{1}{\sqrt{1+n^2/\mathrm{R}^2}}. \quad . \quad . \quad . \quad . \quad (36)$$

If W″ be another wave which traverses the region under consideration, and e^{-a} is its factor of attenuation while so doing, then the amplitude of its received E.M.F. will be impressed with a modulation of the same frequency $n/2\pi$ and having a coefficient M″ where approximately †

$$\mathrm{M}''=\alpha\mathrm{M}'. \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

The quantity α is proportional to ν_0, and is commonly of the order of unity.

We may suppose, with little error, that the collisions involvea constant coefficient of restitution and a constant mean free path, and so the relations (32) and (33) of the previous communication‡ may be adopted, namely,

$$\eta=\tfrac{1}{2}a\mathrm{G}(\nu^2-\nu_1{}^2), \quad . \quad . \quad . \quad . \quad . \quad (38)$$

and

$$\mathrm{R}=\mathrm{G}\nu_0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

† A proof of this statement is similar to that given on p. 955.
‡ V. A. Bailey, *loc. cit.*

The above formulæ (35) and (37) then lead to the following result,

$$M''/M = \alpha\delta g_0/G, \quad . \quad . \quad . \quad . \quad . \quad (40)$$

where g_0 is the ratio of the energy η_0, lost at a collision, to the kinetic energy $\frac{1}{2}mu_0^2$ possessed by the colliding electron when the steady state exists which corresponds to the force Z_0.

We have also from (36) and (39)

$$\delta = \frac{1}{\sqrt{1+n^2/G^2\nu_0^2}}. \quad . \quad . \quad . \quad . \quad . \quad (41)$$

Finally, with the help of (31) and (38), we may express the relation (40) in the following form,

$$\frac{M''}{M} = \frac{\alpha\delta w_0}{w_0+GT}, \quad . \quad . \quad . \quad . \quad . \quad (42)$$

where T is the mean energy of agitation of an electron in the absence of electric fields.

We will now apply these results to the observations made at Eindhoven, by van der Pol and van der Mark†, on the modulation impressed by the wave from Luxembourg (of frequency 230 kc.s.) on the wave received from Beromünster (556 kc./s.).

With Luxembourg working on full power (presumably at its official rating of 150 kilowatts), and with modulations of different frequencies but the same coefficient 0·8, they obtained values of the impressed modulation M″, which lie about a smooth curve ‡.

This curve agrees closely with a curve whose equation is

$$M'' = \frac{0{\cdot}086}{\sqrt{1+(n/780)^2}},$$

i. e.,

$$M'' = 0{\cdot}086\,\delta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

where δ is given by (41) with G assigned the value appropriate to air and ν_0 assigned the value 3×10^5. Thus the present theory of interaction and the original theory are both in close agreement with observation as far as the acoustic distortion δ of the impressed modulation is concerned.

† U. R. S. I. Report dated 10th September, 1934. Also given in *Tijdschrift Ned. Radio Genootschap*, vii. pp. 12–17 (March 1935).

‡ V. A. Bailey and D. F. Martyn, ' Nature,' cxxxv. p. 585 (13th April, 1935).

To compare the absolute values of M″ which are given by the theory and by observation, we will now make the following appropriate substitutions in the formulæ (29) and (42):

$$Z=1{\cdot}73\times10^{-4}\text{ volts/cm.},\quad \omega=1{\cdot}45\times10^{6},\quad M=0{\cdot}8,$$

$$\nu_0=3\times10^{5},\quad G=2{\cdot}6\times10^{-3},\quad T=5{\cdot}8\times10^{-14}.$$

The values of G and T correspond to air at 15° C., and the value of Z is derived from the value 2×10^{-4}, which was previously estimated† to correspond to a radiated power of 200 kilowatts.

We thus find $w_0=1{\cdot}94\times10^{-17}$ and so

$$M''=0{\cdot}091\alpha\delta. \quad . \quad . \quad . \quad . \quad . \quad (44)$$

The value of α, the exponent of attenuation of the sky wave W″ which travelled from Beromünster to Eindhoven, may be approximately estimated by means of Martyn's formula ‡, which is essentially as follows,

$$\alpha=0{\cdot}73\cos i_0\left(\frac{0{\cdot}57}{\omega''\cos i_0}\right)^{1/5}, \quad . \quad . \quad . \quad (45)$$

where ω'' is this wave's angular frequency and i_0 is its angle of incidence on the E-layer.

Since the distance between Beromünster and Eindhoven is about 550 kilometres we have $\tan i_0=2{\cdot}75$, and so $\cos i_0=0{\cdot}34$; since also for Beromünster $\omega''=3{\cdot}5\times10^{6}$, it therefore follows that $\alpha=1{\cdot}31$.

Consequently by (44) the theoretical value of the impressed modulation is given by

$$M''=0{\cdot}119\,\delta.$$

On comparing this value of M″ with the observed value given in (43) it will be seen to exceed the latter by about 40 per cent. This excess is in part due to our ignorance of the exact angular distribution of the radiation from Luxembourg and of the attenuation of this radiation in the E-layer. Another possible cause of the excess is that Luxembourg's wave may not have penetrated to the top of the trajectory of Beromünster's wave. A reduction

† *Cf.* V. A. Bailey and D. F. Martyn, Phil. Mag., *loc. cit.* p. 379.

‡ D. F. Martyn, Commonwealth Bulletin No. 88, Report No. 7, or Proc. Phys. Soc. vol. xlvii. p. 334.

of about 20 per cent. in the assumed value of Z would make the theoretical value of M″ equal to the observed.

We may therefore conclude that, in regard also to the absolute magnitude of the impressed modulation, the present theory is in good agreement with the observations of van der Pol and van der Mark.

The original theory, on the other hand, provided for the absolute magnitude only a lower limit, this limit being about one tenth part of the observed value.

5. *The Modulation impressed by a Gyro-Wave.*

We will consider here the effects caused by a wave whose angular frequency ω is nearly equal to the angular gyro-frequency Ω, and for convenience will name such a wave a gyro-wave and its radiator a gyro-station.

On comparing the results in Tables I. and II. for the gyro wave-length λ_H, it is seen that we need consider primarily only the extraordinary component of the two into which a plane gyro-wave divides on entering the ionosphere. So in the formula (24) we set $\overline{U}=\overline{U}_e a_s$, where $\overline{U}_e$ is the mean energy flux density at the boundary of the ionosphere due to the extraordinary wave alone, and

$$\log a_s = -2\int_0^s \kappa . ds, \quad . \quad . \quad . \quad . \quad . \quad (46)$$

where s is the distance traversed by the wave, measured from its entry into the ionosphere.

In a weakly ionized medium the relation of $\overline{U}_e$ to the energy flux density $\overline{U}_p$ of the original wave plane is given by the following formulæ, which are established in the Appendix II. :

$$\frac{\overline{U}_e}{\overline{U}_p} = \frac{\beta\mu_e}{\mu_e^2+\chi_e^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

where

$$\beta = \left|\frac{R_p-R_o}{R_e-R_o}\right|^2 \cdot \frac{1+|R_e|^2}{1+R_\mu}, \quad . \quad . \quad . \quad (48)$$

and the subscripts p, o, and e refer respectively to the original plane wave and to its ordinary and extraordinary components.

Hence, for the extraordinary wave the formula (24) takes the following form,

$$\bar{w}_e = \tau Z^2 \beta \gamma a_e, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

where

$$\gamma = \frac{2\mu_e \chi_e}{\nu\omega_0(\mu_e^2+\chi_e^2)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

We will now show that the resonance manifested in Table I. for a particular direction of propagation is, under certain conditions, true for all directions.

On eliminating a between (14) and (19) and setting

$$\left.\begin{aligned} x &= -\omega_0(1+\zeta^{-1}), \quad & y &= \omega-\omega_0-i\nu, \\ z &= \Omega^2-y^2, & r &= y-\tfrac{1}{2}\Omega_T^2 y^{-1}, \end{aligned}\right\} \quad . \quad . \quad (51)$$

we obtain

$$x(x-2r)=z. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (52)$$

In the E-layer at night we have

$$\nu << \Omega, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (53)$$

and throughout most of its lower part we have also $N << 3\times 10^4$. So when ω is of the same order as Ω, namely about 10^7, the following relations also hold true :

$$\nu << \omega, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (54)$$

$$\left.\begin{aligned} \omega_0 << \omega, \quad & \omega_0 << \Omega, \\ p_0^2 << \Omega^2. & \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (55)$$

From equation (52) we deduce for the extraordinary wave that

$$x = r-\sqrt{r^2+z},$$

and so

$$\frac{1}{x} = \frac{r+\sqrt{r^2+z}}{-z}.$$

Since under the conditions (53) and (55) we also have

$$|z| < |r^2|,$$

it follows that

$$\frac{1}{x} \doteqdot \frac{2r+z/2r}{-z}. \quad . \quad . \quad . \quad . \quad . \quad (56)$$

By the definitions of x and ζ we have

$$1-\frac{1}{c^2q^2}=-\frac{\omega_0}{x},$$

and so it follows from (56) that

$$c^2q^2=\frac{z}{z-\omega_0(2r+z/2r)}. \quad \ldots \quad (57)$$

From (51) we have

$$\begin{aligned} z&=\Omega^2-\omega^2+2\omega(\omega_0+\iota\nu)-(\omega_0+\iota\nu)^2 \\ &=a+\iota b, \end{aligned}$$

where, on taking (55) into account,

$$\left.\begin{aligned} a&\doteq\Omega^2-\omega^2+2\omega\omega_0+\nu^2, \\ b&\doteq 2\omega\nu. \end{aligned}\right\} \quad \ldots \quad (58)$$

Also from (51), (53), (54), and (55) we find

$$\begin{aligned} r&\doteq\omega-\iota\nu-\frac{1}{2}\frac{\Omega_1^2}{\omega}\left(1-\frac{\iota\nu}{\omega}\right) \\ &=s+\iota t, \end{aligned}$$

where

$$\left.\begin{aligned} s&=\omega-\frac{1}{2}\frac{\Omega_1^2}{\omega^2} \\ t&=-\nu\left(1+\frac{1}{2}\frac{\Omega_1^2}{\omega^2}\right), \end{aligned}\right\} \quad \ldots \quad (59)$$

and, by (54),

$$|t|<<|s|.$$

Therefore

$$\begin{aligned} 2r+z/2r&=2s+2\iota t+(a+\iota b)\frac{1}{2s}\left(1-\frac{\iota t}{s}\right) \\ &=l+\iota m, \end{aligned}$$

where

$$\left.\begin{aligned} l&=2s+\frac{a}{2s}+\frac{bt}{2s^2}, \\ m&=2t+\frac{b}{2s}-\frac{at^2}{2s^2}. \end{aligned}\right\} \quad \ldots \quad (60)$$

Hence

$$z-\omega_0(2r+z/2r)=c+\iota d,$$

where

$$\left.\begin{aligned} c &= a-\omega_0 l, \\ d &= b-\omega_0 m. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (61)$$

From (57) we therefore have

$$c^2q^2 = \frac{a+ib}{c+id}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (62)$$

On setting in this equation

$$b/a=\tan\phi, \quad d/c=\tan\psi, \quad \chi/\mu=\tan\rho, \quad . \quad (63)$$

where by definition $cq=\mu-i\chi$, it leads to the following result:

$$(e^{-i\rho})^2 = e^{i\phi}/e^{i\psi}.$$

Therefore

$$\sin 2\rho=\sin(\psi-\phi).$$

If in the formula (50) we now set

$$S = \frac{2\mu\chi}{\mu^2+\chi^2},$$

it becomes

$$\gamma=S/\nu\omega_0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (64)$$

where

$$S = \frac{2(\chi/\mu)}{1+(\chi/\mu)^2} = \frac{2\tan\rho}{1+\tan^2\rho} = \sin 2\rho,$$

i. e.,

$$S=\sin(\psi-\phi). \quad . \quad . \quad . \quad . \quad . \quad (65)$$

We will now study the variation of γ with ω as the latter is varied around the value Ω.

Accordingly we set

$$\left.\begin{aligned} \omega &= \Omega(1+\epsilon), \\ |\epsilon| &< 1, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (66)$$

where

in the formulæ (58), (59), (60), and (61), and so obtain the following approximations:

$$a=a_0-2\epsilon\Omega^2, \qquad b=b_0+2\epsilon\nu\Omega, \quad . \quad . \quad (67)$$

where $a_0=\nu^2+2p_0^2, \qquad b_0=2\nu\Omega. \quad . \quad . \quad . \quad (68)$

$$s=f\Omega(1+\epsilon h), \qquad t=-\nu(g-\epsilon\sin^2\theta),$$

where $f=1-\frac{1}{2}\sin^2\theta, \qquad g=1+\frac{1}{2}\sin^2\theta, \quad h=g/f.$

$$l=2f\Omega+\epsilon\Omega(2g-f^{-1}),$$
$$m=-\nu(2g-f^{-1})+\epsilon\nu(2\sin^2\theta+f^{-1}).$$

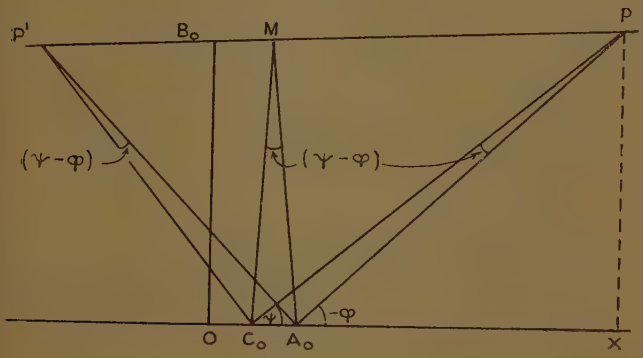
$$c=c_0-2\epsilon\Omega^2, \qquad d=b_0+2\epsilon\nu\Omega, \quad . \quad . \quad . \quad (69)$$

where
$$c_0=\nu^2+p_0^2\sin^2\theta. \quad . \quad . \quad . \quad . \quad . \quad (70)$$

The variation of $(\psi-\phi)$ is very conveniently shown by means of the diagram in fig. 1.

In this diagram we have $OA_0\perp^r OB_0$, $OA_0=a_0$, $OB_0=b_0$, $OC_0=c_0$, $OX=2\epsilon\Omega^2$, and $P'B_0P$ is a straight line of slope equal to ν/Ω.

Fig. 1.



By reason of the relations (67) and (69) we have $A_0X=OX-OA_0=-a$, $C_0X=-c$, and, if X be the projection of P, also

$$XP=OB_0+OX(\nu/\Omega)=b_0+2\epsilon\nu\Omega=b=d.$$

Hence by (63)

$$\angle PA_0X=-\phi, \quad \angle PC_0X=-\psi,$$

and so

$$\angle A_0PC_0=\psi-\phi.$$

This is indicated on the diagram.

S nce the slope of the line $P'B_0P$ is always finite and positive, and since A_0 and C_0 must always lie to the right of O, it follows that this line can never pass between A_0 and C_0. Hence we can always find a point M on it, such that the circle A_0MC_0 touches this line.

A simple argument then shows that $\angle A_0MC_0$ is greater than $\angle A_0PC_0$, when P is any point on the line other than M.

We thus see that as P moves towards B_0 and beyond it to P′, *i. e.*, as ω varies about Ω, the value of $(\psi-\phi)$ passes through a maximum ; hence by (64) the value of S also passes through a maximum.

On account of the relations (53) and (55) the quantity $|\ 2\epsilon\Omega^2\ |$ may attain values large compared with $\nu^2+2p_0^2$, while yet allowing $|\ \epsilon\ | << 1$ as required by (66). Hence the point X may be allowed to move to distances from O large compared with OA_0 and OC_0. Since also by (53) the slope of $P'B_0P$ is small, we conclude that $(\psi-\phi)$, and therefore S, may attain values small compared with its maximum value without the quantity $(1+\epsilon)$ changing much, *i. e.*, without ω_0 changing much from the value p_0^2/Ω.

Hence by (64) we conclude that γ passes through a notable maximum value as ω passes through a value very near to Ω, namely (approximately),

$$\Omega(1+B_0M/2\Omega^2), \quad i.\,e., \quad \Omega\left(1+\frac{\nu^2+gp_0^2}{2\Omega^2}\right).$$

The corresponding variation of the coefficient of attenuation a_s cannot be determined without assigning a structure to the E-layer, such, for example, as that adopted below for the calculation of a_s when $\omega \doteqdot \Omega$. This problem will be considered in detail elsewhere. Meanwhile we may conclude that where at the gyro-frequency a_s is not much less than unity, there the quantity γa_s manifests a notable degree of resonance as ω passes through the value Ω.

It will now be shown that β, the remaining factor in $\bar{w}_e$ which depends on ω, does not change much as ω varies around the value Ω by amounts which produce large changes in γ.

The equation (14) may, by (19), be written as

$$-\Omega_L R = \nu+\iota\omega+\iota\omega_0/(c^2q^2-1).$$

But by (62) we have

$$\frac{1}{c^2q^2-1} = \frac{c+\iota d}{(a-c)+\iota(b-d)} = \frac{c+\iota d}{a_0-c_0} = \frac{c+\iota d}{2fp_0^2}.$$

Hence

$$R_e = -\Omega_L^{-1}\{\nu + \iota\omega + \iota(c_0 - 2\epsilon\Omega^2)/2f\omega - (b_0 + 2\epsilon\nu\Omega)/2f\omega\}$$

$$= \frac{1}{\iota \cos\theta}\{1 + \epsilon(1 - f^{-1})\},$$

i. e.,

$$\left.\begin{aligned} R_e &= R_r(1 - C\epsilon), \\ \text{where} \quad R_r &= \frac{1}{\iota\cos\theta} \quad \text{and} \quad C = \frac{\sin^2\theta}{1 + \cos^2\theta}. \end{aligned}\right\} \quad . \quad . \quad (71)$$

Also, since R_o and R_e are the two roots of the quadratic equation (19), we must have $R_o R_e = 1$.

Hence

$$R = R_r^{-1}(1 + C\epsilon). \quad . \quad . \quad . \quad . \quad . \quad (72)$$

Thus R_o and R_e are pure imaginaries.

Since R_p in the formula (48) is real it follows that in that equation

$$|R_p - R_o|^2 = R_p^2 + l_\theta^2(1 + C\epsilon)^2 = R_p^2 + l_\theta^2 + 2l_\theta^2 C\epsilon,$$

where $l_\theta = \cos\theta$.

Also

$$|R_e - R_o|^2 = \{l_\theta^{-1}(1 - C\epsilon) + l_\theta(1 + C\epsilon)\}^2 = (l_\theta^{-1} + l_\theta)^2(1 - 2C^2\epsilon),$$

and $$1 + |R_e|^2 = 1 + l_\theta^{-2}(1 - C\epsilon)^2 = 1 + l_\theta^{-2} - 2l_\theta^{-2}C\epsilon.$$

Then the formula (48) becomes

$$\beta = \beta_r \frac{1 + a'\epsilon}{1 - b'\epsilon} \cdot \frac{1 - c'\epsilon}{1 + R_p^2},$$

where

$$a' = \frac{2l_\theta^2 C}{R_p^2 + l_\theta^2}, \quad b' = 2C^2, \quad c' = \frac{2C}{1 + l_\theta^2},$$

$$\beta_r = \frac{1 - \sin^2\theta\sin^2\psi}{1 + \cos^2\theta}, \quad . \quad . \quad . \quad . \quad . \quad (73)$$

and ψ is defined in Appendix II.

Hence

$$\beta = \beta_r(1 + B\epsilon), \quad . \quad . \quad . \quad . \quad . \quad (74)$$

where

$$B = a' + b' - c' = 2l_\theta^2 C\left(\frac{1}{R_p^2 + l_\theta^2} - \frac{1}{1 + l_\theta^2}\right)$$

$$= 2C\left(\frac{1}{1 + l_\theta^2} - \frac{1}{1 + l_\theta^2 R_p^{-2}}\right).$$

Since $0 \leqslant l_\theta^2 \leqslant 1$, $R_p^{-2} \geqslant 0$, it follows by a simple argument that $|B| \leqslant 2$, and so β does not change much as ω varies round Ω.

We thus conclude that where a_s is not much less than unity the quantity w passes through a notable maximum as ω passes through the value Ω.

Since this resonance arises from the factor γ, we will now derive a formula for γ which allows the shape of the resonance curve to be determined. In order to obtain a simple formula, without, however, restricting our discussion of the principal effects caused by a gyro-wave, we adopt the following limitation :

$$\omega_0 << \nu. \quad . \quad . \quad . \quad . \quad . \quad . \quad (75)$$

In that case we have

$$p_0^2 << \nu\Omega, \text{ and so } C_0A_0 << OB_0.$$

Thus $(\psi - \phi)$ is then always small, and so

$$S = \sin(\psi - \phi) \doteqdot \tan(\psi - \phi) = \frac{ad - bc}{ac + bd} = \frac{a - c}{d + ac/b}.$$

But by (67) to (70)

$$a - c = a_0 - c_0 = 2f \sin^2\theta,$$

$$ac/b = (a_0 - 2\epsilon\Omega^2)(c_0 - 2\epsilon\Omega^2)/(b_0 + 2\epsilon\nu\Omega)$$

$$= \frac{1-\epsilon}{2\nu\Omega}\{a_0c_0 - 2\epsilon\Omega^2(2\nu^2 + 2gp_0^2) + 4\epsilon^2\Omega^4\}$$

$$= \frac{1}{2\nu\Omega}\{a_0c_0 - 4\epsilon\Omega^2(\nu^2 + gp_0^2) + 4\epsilon^2\Omega^4\},$$

$$d = 2\nu\Omega + 2\epsilon\nu\Omega.$$

Hence

$$d + ac/b = 2\nu\Omega + 2\nu\Omega\epsilon\left\{1 - \left(1 + g\frac{p_0^2}{\nu^2}\right)\right\} + \epsilon^2\frac{\Omega^2}{\nu^2}2\nu\Omega$$

$$= 2\nu\Omega\{1 - \epsilon(gp_0^2/\nu^2) + \epsilon^2(\Omega^2/\nu^2).$$

Therefore

$$\frac{S}{S_r} = \frac{1}{1 - p\epsilon + q\epsilon^2},$$

where

$$p = gp_0^2/\nu^2, \quad q = \Omega^2/\nu^2, \quad S_r = fp_0^2/\nu\Omega.$$

Since by (64) we have

$$\gamma = S_r\Omega/\nu p_0^2 = (1 + \epsilon)S/S_r,$$

it therefore follows that

$$\gamma=\gamma_r X, \quad . \quad . \quad . \quad . \quad . \quad . \quad (76)$$

where

$$\left.\begin{aligned} X&=\frac{1}{1-C_1\epsilon+C_2\epsilon^2}, \\ C_1&=1+gp_0^2/\nu^2, \quad C_2=\Omega^2/\nu^2, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (77)$$

$$\gamma_r=f/\nu^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (78)$$

$$\left.\begin{aligned} f&=1-\tfrac{1}{2}\sin^2\theta, \\ g&=1+\tfrac{1}{2}\sin^2\theta. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (79)$$

In Table III. are shown values of X calculated by means of (77) for the following circumstances :

$$N=200, \quad \nu=6\times10^5, \quad \Omega=10^7, \quad \theta=45^\circ;$$

i. e.,

$$C_1=4{\cdot}8, \quad C_2=280.$$

TABLE III.

ϵ	$-0{\cdot}2$	$-0{\cdot}1$	$-0{\cdot}05$	0	0·05	0·1	0·2
X	0·076	0·234	0·515	1	0·685	0·301	0·089

We see from this table that a variation of about 5 per cent. in the value of ω on either side of the value Ω may cause $\overline{w}_e$ to become nearly halved.

We will now consider the situations which are first encountered by a gyro-wave on entering the E-layer, namely, those in which the conditions (53), (54), (55), and (75) exist. For Europe these correspond to the conditions

$$3\times10^5<\nu<10^6, \quad N<<10^3. \quad . \quad . \quad . \quad (80)$$

In fig. 1 the vectors $\overset{\smile}{PA}_0$ and $\overset{\smile}{PC}_0$ correspond respectively to the numerator and denominator of the formula for c^2q^2 in (62). Hence cq is a complex operator which turns a vector through an angle $\frac{1}{2}(\phi-\psi)$ and multiplies it by Q, where

$$Q=\sqrt{\overset{\smile}{PA}_0/\overset{\smile}{PC}_0}.$$

We have, therefore,

$$\mu-\iota\chi=Qe^{\frac{1}{2}\iota(\phi-\psi)},$$

and so

$$\mu=Q\cos\tfrac{1}{2}(\psi-\phi), \quad \chi=Q\sin\tfrac{1}{2}(\psi-\phi).$$

On account of the small slope of $P'B_0P$ and the relation $C_0A_0 << OB_0$, it follows that $(\psi - \phi)$ is always small, and $Q \doteqdot 1$.

Hence in the lower part of the E-layer

$$\mu \doteqdot 1, \quad \chi = \tfrac{1}{2}S. \qquad (81)$$

Thus

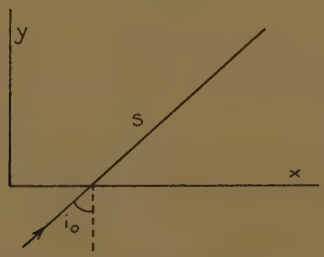
$$\kappa = \omega\chi/c = \Omega(1+\epsilon)S/2c,$$

and so

$$\left.\begin{aligned} \kappa &= \kappa_r X, \\ \text{where} \quad \kappa_r &= 0{\cdot}0535 f N/\nu. \end{aligned}\right\} \qquad (82)$$

The formulæ (82) show that *near resonance* the extraordinary wave has κ inversely proportional to ν, whereas at frequencies well removed from Ω (with low values of N) both the ordinary and extraordinary waves have κ

Fig. 2.



increasing or decreasing with ν. Martyn's absorption curves are clearly in agreement with this conclusion †.

We now proceed to examine the events which occur as the gyro-ray travels into the ionosphere along a path s, and for convenience we here confine the discussion to that of latitudes sufficiently distant from the equator to justify our taking the earth's magnetic field as approximately vertical.

The coordinate system will be taken as in fig. 2 with the x-axis horizontal, the y-axis vertical, and the (x, y) plane containing the trajectory of the ray.

The plane $y=0$ is selected to lie at a height (about 90 km. up) where the ray first begins to be appreciably affected by the ionized atmosphere at night.

† A similar result is obtained by Nicholls and Schelling for longitudinal propagation.

With our condition $N << 10^3$ we have $\mu \doteq 1$, and so, excepting when i_0 is nearly equal to $\pi/2$ (which may here be excluded for other reasons), we may take the path s to be a straight line in the regions under consideration, *i. e.*, we have,

$$s = y \sec i_0.$$

Martyn † has stated that the propagation of medium radio waves at night is consistent with the following assumptions about the variation of ν and N with the height y :

$$\left.\begin{aligned} &\nu = \nu_0 e^{-ay}, \quad \nu_0 = 1{\cdot}1 \times 10^6, \quad a = 10^{-6}, \\ \text{and} \quad &N = N_0 e^{by}, \quad N_0 = 1, \quad\quad b = 10^{-5}. \end{aligned}\right\} \quad . \quad . \quad (83)$$

For our purpose we may provisionally adopt these assumptions.

Then our formulæ (82), (76), and (46) yield respectively

$$\left.\begin{aligned} &\kappa = A f e^{cy} X, \\ &A = 4{\cdot}86 \times 10^{-8}, \quad c = 1{\cdot}1 \times 10^{-5}, \end{aligned}\right\} \quad . \quad . \quad (84)$$

$$\left.\begin{aligned} &\gamma = B f e^{2ay} X, \\ &B = 8{\cdot}3 \times 10^{-13}, \quad 2a = 2 \times 10^{-6}, \end{aligned}\right\} \quad . \quad . \quad (85)$$

and

$$\left.\begin{aligned} &\log a_s = -C f . \sec i_0 (e^{cy} - 1) X, \\ &C = 8{\cdot}8 \times 10^{-3}. \end{aligned}\right\} \quad . \quad . \quad . \quad (86)$$

In these formulæ by (79) we may here set

$$f = \tfrac{1}{2}(1 + \cos^2 i_0). \quad . \quad . \quad . \quad . \quad . \quad (87)$$

The variation of $\bar{w}_e$ with y may therefore be determined by means of (49), (85), and (86) ‡.

The factor f in (85) must lie between 0·62 and 1, and the factor $f \sec i_0$ in (86) lies between 1 and 1·25, for all values of i_0 not exceeding 60°. Hence the values of $\bar{w}_e$ corresponding to $i_0 = 45°$ are fairly representative of the values for all directions of propagation at an angle with the horizon exceeding 30°. Such values are given in Table IV., which corresponds to a wave producing a field-intensity of 5 millivolts/m. just below the E-layer of the ionosphere, to $\epsilon = 0$, and to an assumed value of $\frac{1}{2}$ for

† D. F. Martyn, *loc. cit.*

‡ With a strong wave the value of ν will depend in part on Z, and then another relation is needed, namely (89).

the factor β in the formula (49). As equation (73) shows, the factor β cannot be estimated without a knowledge of the value of ψ, and this value will depend on the individual radiating station under consideration. For our present purpose it is sufficient to adopt for β an average value of $\frac{1}{2}$, as this is the average value obtained from formula (73) on the simple assumption that all values of ψ are equally likely.

The maximum value of w_e occurs when $\frac{d(a_x\gamma)}{dy}=0$, *i. e.*, when $\kappa=a\cos i_0=7{\cdot}1\times10^{-7}$, or $y=2{\cdot}7$ km.

From the second row in Table IV. it will be seen that w_e becomes relatively negligible for values of y exceeding 5·5 km. and values of N exceeding 200 electrons/c.c.

TABLE IV.

$i_0=45°$, Z=5 mV/m., W=10 kw. or less, $\epsilon=0$.

y kms....	0	1·5	2·7	4·0	4·5	5	5·5	6·0
a_x.....	1	·96	·84	·47	·265	·103	·020	·001
$\bar{w}_e\times10^{18}$	2·2	2·8	3·2	2·3	1·5	·6	·13	·01
N	1	4·5	14·9	54·6	90	150	244	400
$\nu\times10^{-5}$	11	9·5	8·4	7·4	7·0	6·6	6·3	6·0

Thus the approximations we have made, involving the condition $N<<1000$, are adequate to our present purpose.

Also it is clear from this table that $\bar{w}_e$ is approximately constant and equal to 2×10^{-18} ergs at all points within a slab of the ionosphere, which is about 4·5 km. thick.

With an emitter behaving approximately like a vertical Hertzian oscillator, the maximum value of Z corresponds to that direction for which $i_0=45°$, *i.e.*, to the conditions involved in Table IV. For, if W watts be the radiated power, then

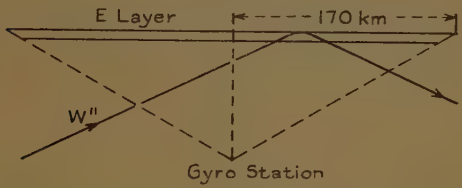
$$Z=10^{-6}W\sin i\cos i,\ \text{volts/cm.}$$

This formula indicates that 10 kW is the value of the power W needed to make the electric force Z equal to 5 mV/m at the lower boundary of the E-layer. Since actual radiating aerials usually have notable horizontal components the value of W needed may be somewhat less.

Also it may be assumed, without serious error, that with such aerials Z is constant over the part of the E-boundary which corresponds to the range of values of i_0 lying between 0° and 60°.

Hence $\bar{w}_e$ may be taken to be constant in the part of the above-mentioned 4·5 km. slab which is contained within a cone of semi-vertical angle 60°. This region is illustrated in fig. 3.

Fig. 3.



The proportional change of ν ultimately produced by the wave may now be determined as follows :—

If ν_1 be the initial value of ν and ν_0 be the steady value † ultimately produced by the wave, then, since the equations (31) and (38) in section 4 are also applicable here, we have

$$\tfrac{1}{2}a\mathrm{G}(\nu_0^2-\nu_1^2)=\bar{w}_e, \quad . \quad . \quad . \quad . \quad . \quad (88)$$

or

$$\frac{\nu_0}{\nu_1}=\sqrt{1+\frac{\bar{w}_e}{\mathrm{GT}}}, \quad . \quad . \quad . \quad . \quad . \quad (89)$$

where

$$\mathrm{G}=2{\cdot}6\times10^{-3} \quad \text{and} \quad \mathrm{T}=5{\cdot}8\times10^{-14} \text{ ergs at } 15^\circ \text{ C.}$$

Assuming that the temperature in the slab is not very different from 15° C., and setting $\bar{w}_e=2\times10^{-18}$ we thus find that

$$\nu_0/\nu_1=1{\cdot}0067.$$

Hence

$$\Delta\nu/\nu=0{\cdot}67 \text{ per cent.}$$

If the wave be modulated at a musical frequency $n/2\pi$, *i. e.*, if

$$\mathrm{Z}=\mathrm{Z}_0(1+\mathrm{M}\sin nt), \text{ where } \mathrm{M}<1,$$

† Not to be confused with the symbol ν_0 in (83).

then, as in section 4, the ultimate variation of ν is given by the permanent part of the integral of the differential equation (33), when we set

$$\Delta\bar{w}=2\mathrm{M}\bar{w}_e \sin nt-\tfrac{1}{2}\mathrm{M}^2\bar{w}_e \cos 2nt.$$

Since ν_1 differs from ν_0 by only 0·67 per cent. we may suppose that ν does not differ by more, and so with our present example we may for the above equation suppose that $\bar{w}_e$ is constant in time.

Thus the steady variation of ν is given approximately by the particular integral (34).

Hence ν is modulated at the frequency $n/2\pi$ and with a coefficient of modulation M′ given by (35), *i.e.*, by the formula

$$\mathrm{M}'\doteqdot \frac{\mathrm{M}\bar{w}_e\delta}{\dot{\mathrm{G}}\mathrm{T}+\bar{w}_e}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (90)$$

With the numerical values already adopted and with M=0·5 (as a typical value) we find

$$\mathrm{M}'=6{\cdot}6\times10^{-3}\delta. \quad . \quad . \quad . \quad . \quad . \quad . \quad (91)$$

Near the top of the slab $\nu_0=7\times10^5$, so at the musical frequency 256 (Middle C) $\delta=0{\cdot}75$, and, consequently,

$$\mathrm{M}'=0{\cdot}5 \text{ per cent.}$$

We may now consider some conditions under which this modulation of ν can cause observable effects on another wave W″ passing through our slab.

The most favourable conditions are those which cause the wave W″ to suffer its absorption almost entirely in the slab. Corresponding to each value of the angle of incidence i_0, the shortest wave-length λ'' which W″ can then have is that which causes the top of its trajectory to lie in the plane $y=4{\cdot}5$ km., and so the highest value of the angular frequency ω'' is given by the formulæ †

$$\mu=\sin i_0, \quad \mu^2=1-\frac{4\pi e^2\mathrm{N}}{m(\omega''^2+\nu^2)},$$

where

$$\mathrm{N}=90, \quad \nu=7\times10^6.$$

† For the explanation of our adoption of the formula for μ reference may be made to D. F. Martyn, Commonwealth Bulletin, no. 88, Radio Research Board, Report no. 7, pp. 16 and 17.

Therefore

$$\omega''^2 + 4{\cdot}9 \times 10^{11} = 2{\cdot}88 \times 10^{11} \sec^2 i_0,$$

or

$$\omega'' = 10^5 \sqrt{28{\cdot}8 \sec^2 i_0 - 49}. \quad . \quad . \quad . \quad (92)$$

We thus obtain the results set out in the first three rows of Table V. The corresponding attenuation coefficients, given in the fourth row, are calculated by means of the formula (45).

The modulation M'' impressed on W'' is shown in the fifth row and has been calculated as follows :—

Since κ at all points on the path is proportional to ν, which has nearly the same coefficient of modulation M'

TABLE V.

ω''=maximum suitable angular frequency ;

λ''=minimum suitable wave-length.

i_0 degrees	45	60	70	80
$\omega'' \times 10^{-5}$	2·9	8·1	14·1	30·2
λ'' metres	6500	2330	1340	625
$e^{-\alpha}$	·018 †	·085 †	·197	·444
M'' per cent.	2·0	1·2	0·8	0·4
D kilometres	200	346	550	1130

everywhere, we must regard α ($=\int \kappa\, ds$) as subject to the same modulation, *i. e.*,

$$\Delta\alpha = \alpha M' \sin(nt - \phi).$$

Hence the received intensity of W'' is subject to the modulation $e^{-\Delta\alpha}$, *i. e.*, to a modulation whose coefficient is given by

$$M'' = \alpha M'. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (93)$$

In the last row are given the corresponding distances D from their source at which the waves W'' are received, the formula used being as follows :

$$D = 200 \tan i_0 \text{ km.}$$

According to the formula (45) α increases as the fifth root of the wave-length λ'', *i. e.*, very slowly ; hence

† These results are only roughly correct, because Martyn's formula (45) is not accurate for long waves.

we may regard the values of M″ shown in Table V. as giving the approximate order of magnitude of the modulation impressed on a wave whose length exceeds the corresponding number in the third row of the table.

These modulations are of an order from four to ten times greater than that which can usually be observed.

We may now summarize our conclusions as follows :—

At distances beyond 350 km. waves of lengths exceeding 2300 metres can have observable modulations caused by gyro-interaction † if their sky waves are reflected in some part of the slab shown in fig. 3. At distances exceeding 550 km. waves of shorter lengths down to 1300 metres and at distances exceeding 1100 km. waves of lengths as low as 600 metres, can have similar observable modulations.

Thus under favourable conditions, *e. g.*, with freedom from atmospherics and on a suitable unmodulated carrier wave, the modulations impressed by a radio station with a power as low as 1 *k*W may be observed, provided that this station's frequency is close to the value of $He/2\pi m$, where H is the total terrestrial magnetic field intensity at points of the E-layer roughly in the neighbourhood of the station.

The above considerations then suggest that there is some likelihood of interaction being observed on suitable waves whose points of ionospheric reflexion lie within a radius of 200 km. of at least one of the stations ‡ given in Table VI.

Besides gyro-stations there are a number of other stations § the first harmonics of whose waves are near the local gyro-frequencies and may be sufficiently strong to impress observable modulations on other waves. It would be possible to test whether the harmonic is chiefly responsible for the effects, either by varying the harmonic content of the emitted wave or by slightly varying this wave's frequency. With the second test a large change in the observed impressed modulations would indicate that the harmonic is the main cause.

† *I. e.*, radio interaction caused by a gyro-wave.

‡ *Note added in proof* : This prediction, first published in ' Nature ' on 9th Jan. 1937, has now been confirmed by observation of the interaction caused by Lille on the waves from Hilversum I, Luxembourg, Beromünster, and Strasbourg. In some instances this interaction was still easily observed when Lille's radiated power was 10 *k*W.

§ Such as Prague, Lyons (P.T.T.), Cologne, and North Regional.

The above theory also shows that the acoustic distortion of the modulation impressed by a gyro-wave is about the same as that impressed by a long wave.

In order that the main conclusions about gyro-interaction should not appear to depend solely on the type of electron distribution adopted (namely Martyn's), I have asked Mr. J. M. Somerville to carry out the corresponding calculations for a distribution similar to that

TABLE VI.

Station.	Wave-length, metres.	Power, kw.	λ_{cr}, metres.
Monte Ceneri	257·1	15	246·5
Freiburg	251	5	242
Frankfurt	251	25	241
Lille	247·3	60	236·5
Radio Marconi	245·5	50	238
Cork	241·9	1	234
Saarbrücken	240·2	17	240
Nürnberg	236·8	2	238
Klagenfurt	231·8	5	242
Vorarlberg	231·8	5	242
Malmö	228·7	2·5	228
Kiel	225·6	2	231
Hanover	225·6	2	236
Stettin	225·6	2	232
Dublin	222·6	0·5	232
Los Angeles KECA	209·8	1	210
Nashville WLAC	204·1	5	197
Oklahoma City KOMA	202·7	5	195

postulated by E. V. Appleton and J. A. Ratcliffe †, namely $N \propto y^2$. He finds that the magnitudes of the impressed modulation and of the wave-lengths on which it may be found impressed are of the same order as those we have obtained with Martyn's exponential distribution. Other types of distribution can be conceived which lead not only to similar results, but also to more notable impressed modulations on medium broadcast waves.

The effects caused when ϵ differs from zero will be discussed in another communication.

† Proc. Roy. Soc. A, cxxviii. p. 133 (1930).

In conclusion, I would like to express my indebtedness to Mr. J. M. Somerville, who has checked a large part of the calculations and helped notably in the preparation of Table VI. Such assistance has been made possible throngh research funds supplied by the Carnegie Corporation and by the Commonwealth of Australia.

Appendix I.

In passing through a cylindrical volume of length dx and unit cross-section the wave expends energy on the $N \cdot dx$ contained electrons at the rate $\bar{U}_x - \bar{U}_{x+dx}$, *i. e.*, at the rate $-\dfrac{\partial \bar{U}}{\partial x} dx$, where U is given by the formula (23).

Hence the mean energy expended on one electron per collision is given by

$$\bar{w} = -\frac{1}{N\nu} \frac{\partial \bar{U}}{\partial x}.$$

Since the mean value is taken over the time alone it follows from (23) that

$$\frac{\partial \bar{U}}{\partial x} = -2\kappa U,$$

and so we have

$$\bar{w} = \frac{2\kappa}{N\nu} U.$$

This result † is in complete agreement with (25).

Appendix II.

The sharing of the energy flux of a plane wave between the two elliptic waves into which it divides at the boundary of a homogeneous, non-magnetic ionized medium may be determined as follows :—

With the same coordinate system as that in Section 2 we take $\breve{H}_p$, $\breve{H}_o$, and $\breve{H}_e$ to be the magnetic vectors corresponding respectively to the plane wave and to the ordinary and extraordinary waves into which it divides.

† For a non-homogeneous medium we would have $\kappa + x \dfrac{\partial \kappa}{\partial x}$ in place of κ.

Since the medium is non-magnetic† we may take the magnetic field as varying continuously across the boundary. Hence we have the following relations :

$$H_{py} = H_{oy} + H_{ey}, \quad \ldots\ldots \quad (94)$$

$$H_{pz} = H_{oz} + H_{ez}, \quad \ldots\ldots \quad (95)$$

$$H_{oz} = R_o H_{oy}, \quad \ldots\ldots \quad (96)$$

$$H_{ez} = R_e H_{ey}, \quad \ldots\ldots \quad (97)$$

$$H_{pz} = R_p H_{py}, \quad \ldots\ldots \quad (98)$$

where R_o and R_e are the roots of equation (19) and $R_p = \cot\psi$, where ψ is the angle between z-axis and the vector $\breve{H}_p$.

Elimination of H_{oy} and H_{oz} between (94), (95), and (96) yields

$$H_{pz} - H_{ez} = R_o(H_{py} - H_{ey}).$$

Hence by (97) and (98) we have

$$R_p H_{py} - R_e H_{ey} = R_o H_{py} - R_o H_{ey},$$

and so

$$\frac{H_{ey}}{H_{py}} = \frac{R_p - R_o}{R_e - R_o}.$$

Therefore by (26)

$$\frac{\bar{U}_e}{\bar{U}_p} = \left|\frac{R_p - R_o}{R_e - R_o}\right|^2 \cdot \frac{(1+|R_e|^2)}{(1+|R_p|^2)} \cdot \frac{\mu_e/(\mu_e^2+\chi_e^2)}{\mu_p/(\mu_p^2+\chi_p^2)}.$$

But

$$\mu_p = 1, \quad \chi_p = 0, \quad R_p = \cot\psi, \text{ a real quantity,}$$

and

$$\bar{U}_p = \frac{c}{4\pi}\bar{H}_p^2 = \frac{c}{4\pi}Z^2.$$

Hence

$$\frac{U_e}{\bar{U}_p} = \frac{\beta\mu_e}{\mu_e^2+\chi_e^2},$$

where

$$\beta = \left|\frac{R_p - R_o}{R_e - R_o}\right|^2 \cdot \frac{1+|R_e|^2}{1+R_p^2}.$$

† The diamagnetism which Schrödinger and H. A. Wilson derive for such a medium may be neglected in our case.

Summary.

A fresh investigation is made of the theory of the interaction of radio waves, and this leads to the following results (among others) :—

1. The theory of the interaction caused by powerful long-wave stations is in good agreement with the observations of van der Pol and van der Mark in all essential respects.

2. When a station radiates a wave whose frequency is made to pass through the local gyro-frequency there occurs notable resonance in the radio interaction which may be caused by this station.

3. With frequencies near the gyro-frequency a radiated power of amount as low as 1 kilowatt may suffice to cause *observable* interaction.

4. By means of a gyro-station observations may be made which allow the values of the local terrestrial magnetic force and collision frequency ν to be estimated.

XCIII. *The Measurement of Internal Friction in some Solid Dielectric Materials.* *By* Andrew Gemant *and* Willis Jackson, *Engineering Laboratory, Oxford* *.

I. Introduction.

UNTIL recently internal friction in solids has been discussed analytically on the supposition that it could be treated as equivalent to a viscosity. The equation of motion of an elementary particle of the material has included a "fluid friction" term equal to a constant times the rate of change of unit displacement; the constant is called the internal viscosity, and is supposed to have different values according as the stress is normal or tangential. It follows from this analysis that the amplitude of a vibrating system—for example, torsional oscillations of a rod—should decay exponentially with time, and that the loss of energy per cycle should be proportional to the frequency of vibration.

However, recent experiments have shown that the amplitude does not always decay exponentially with

* Communicated by Prof. R. V. Southwell, F.R.S.

time, and that the energy loss per cycle is sensibly independent of frequency. It is thus clear that the classical conception outlined above must be modified, and, starting with the experimental results, an analysis appropriate to them must be evolved. One possibility is to retain the conception of a frictional force proportional to the velocity but to discard that of a *viscosity constant*, and to replace the previous constant by a factor which is an appropriate function of the frequency (see p. 351 of J. H. C. Thompson, Phil. Trans. Roy. Soc. A, ccxxxi. p. 339 (1933)); another consists in dropping the velocity factor and replacing it by a function of the amplitude of vibration (see A. C. Kimball, 'Proceedings of the 3rd International Congress of Applied Mechanics,' iii. p. 190 (1930)).

It is important to recognize that the difference between these two representations is not merely one of mathematics; it concerns directly the physical mechanism of the frictional damping in solid materials. However, the experimental data available on the subject is very scanty, and insufficient to permit a clarification of this mechanism.

This paper describes some measurements of the loss in solid materials of the dielectric type, namely, ebonite, quartz, glass, Trolitul, and wood. The loss has been measured over a frequency range of about 30 to 1, but the experimental method could not be used for frequencies greater than about 10 cycles/sec. There are two sets of experiments: in one set thin cylindrical rods were subjected to torsional oscillation, and in the other set thin bars were subjected to lateral vibrations. A study was made of the dependence of the damping of the torsional oscillations on the temperature of the material, and these measurements have yielded rather unexpected results. A few measurements have been made on metals, but the experimental technique was not suited to an extensive study of their behaviour.

II. Torsional Vibrations of Thin Cylindrical Rods.

(1) *Classical Theory.*

Although, as foreshadowed in the preceding, the classic or viscous theory of damping proves to be inconsistent with the experimental results described later, it provides a basis for discussion, and will, therefore, be outlined.

The theory is very simple in the case of the torsional oscillation of a thin round shaft, provided that the frequency of the system is small compared with the fundamental frequency of the unloaded shaft. For then no sensible error is made in ignoring the mass of the shaft and in supposing that the angle of twist is proportional to the distance from the point of support. Accordingly the elastic restoring couple which the shaft of length l applies to the mass of moment of inertia I is $\frac{\mu J}{l}.\theta$, where μ is the modulus of rigidity of the material (frequently denoted by C or N), J is the polar moment of the area of cross-section, and θ the angle of twist. In addition, there is a frictional couple produced by a hypothetical shear stress given as the product of a "tangential viscosity constant" κ and the rate of generation of shear-strain, and expressible for the whole shaft in the form $\frac{\kappa J}{l}.\dot{\theta}$. The equation of motion of the loaded shaft is, therefore,

$$I\ddot{\theta}+\frac{\kappa J}{l}.\dot{\theta}+\frac{\mu J}{l}.\theta=0, \quad . \quad . \quad . \quad . \quad (1)$$

of which the solution is

$$\theta=\theta_0 e^{-\frac{\kappa J}{2Il}t}.\cos\sqrt{\frac{\mu J}{Il}}.t. \quad . \quad . \quad . \quad . \quad (2)$$

Hence the damping factor may be written

$$\Delta\equiv\frac{\kappa J}{2Il}=\frac{\kappa\omega^2}{2\mu}, \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where

$$\omega=\sqrt{\frac{\mu J}{Il}}=2\pi \text{ times the natural frequency } f. \quad (4)$$

Expressing Δ in terms of the time t required for the amplitude to decay from θ_1 to θ_2, we have

$$\Delta=\frac{1}{t}\log\frac{\theta_1}{\theta_2},$$

and hence

$$t=\frac{2\mu}{\omega^2\kappa}.\log\frac{\theta_1}{\theta_2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Thus if the frequency of the system is changed, either by altering the length or diameter of the shaft of a given

material or by altering the inertia of the concentrated mass, a straight line should result from plotting t against $1/\omega^2$, t being the time taken for the amplitude to decay in any predetermined ratio.

The magnitude of the quantities μ and κ can be calculated from the experimental data by use of equations (4) and (5) respectively.

Having in mind the subsequent results, it is of interest to consider the rate of energy dissipation during the damped vibration given by equation (1). For a vibration of which the amplitude decays according to the exponential law $e^{-\Delta t}$, the energy diminishes with time as $e^{-2\Delta t}$ and the *relative* loss of energy in time t is $(1-e^{-2\Delta t})$, or $2\Delta t$ if Δ is small. The loss of energy during one cycle, $t=1/f$, taken as a fraction of the initial energy, is therefore, $2\Delta/f$, and, from equation (2), this "specific loss" can be expressed as

$$\delta=4\pi^2 \cdot \frac{\kappa}{\mu} \cdot f, \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and should thus increase linearly with the frequency of vibration.

(2) *The Experimental Method.*

The pendulum was hung from a heavy metal plate screwed to and overhanging the edge of a massive table, the torsion rod being fixed to this plate and to the oscillating load by means of split chucks of brass. To reduce the damping caused by turbulent motion of the surrounding air, the load was formed of turned brass disks centred along the axis of the rod.

Since the purpose of the experiments was to measure the damping due to the torsion shaft, it was important to reduce other causes of damping to a negligible amount. Damping due to air turbulence can be still further reduced by enclosing the whole system in a vacuum. But when this was done no change could be detected in the damping, even when the air-pressure was reduced to 4 mm. of mercury column. Having shown that the air damping was negligible, all further measurements were made in free air. The importance of the damping introduced by the supports is difficult to assess: the discussion of this is best left to the section dealing with experimental results.

The oscillation was started by moving a bar-magnet to and fro near a steel pin fixed in the edge of the disk-shaped mass : the magnet was moved to and fro in a direction tangential to the disk until sufficient amplitude had been generated. To inhibit the simultaneous production of small lateral oscillations, the mass was steadied by a pin and bearing on its underside : the bearing was removed before the damping measurements were made. The amplitude was observed optically by means of a mirror and scale.

To provide means of studying the relation between frictional constant and temperature, the shaft was surrounded by a well-lagged, electrically-heated tube, 2 inches in diameter. The heater wire was wound non-inductively, so that there would be no appreciable magnetic field capable of exciting eddy currents in the oscillating system. The complete assembly is shown diagrammatically in fig. 1. The heating tube was closed at the top by the metal plate supporting the pendulum. It extended 2 inches below the bottom of the shaft and was partially closed at its lower end. Thermocouples were used to measure the temperature at three points near the axis of the oven, and the temperature of the shaft was deemed to be the mean of the temperatures recorded by the thermocouples. Temperature midway along the oven was always slightly higher than the temperature near the ends ; but by increasing the heating current slowly, this difference could be kept less than 3° C. in the worst conditions, which occur at the highest temperatures. Glass and quartz shafts were raised to a temperature of 180° C., but ebonite and Trolitul rods could not be heated above 50° and 60° respectively without becoming too soft to use. Save for one series of measurements on an ebonite shaft 4 mm. diameter, all the shafts had a diameter of 2 mm. The frequency was varied both by changing the length of the shaft and the moment of inertia of the oscillating mass. The practicable range of frequency was about 20 to 1, the highest frequency being 3 cycles/second.

(3) *Experimental Results.*

The frequency of vibration was found to vary as $\frac{1}{\sqrt{Il}}$ (see equation (3)) for shafts of ebonite, Trolitul, quartz,

Fig. 1.

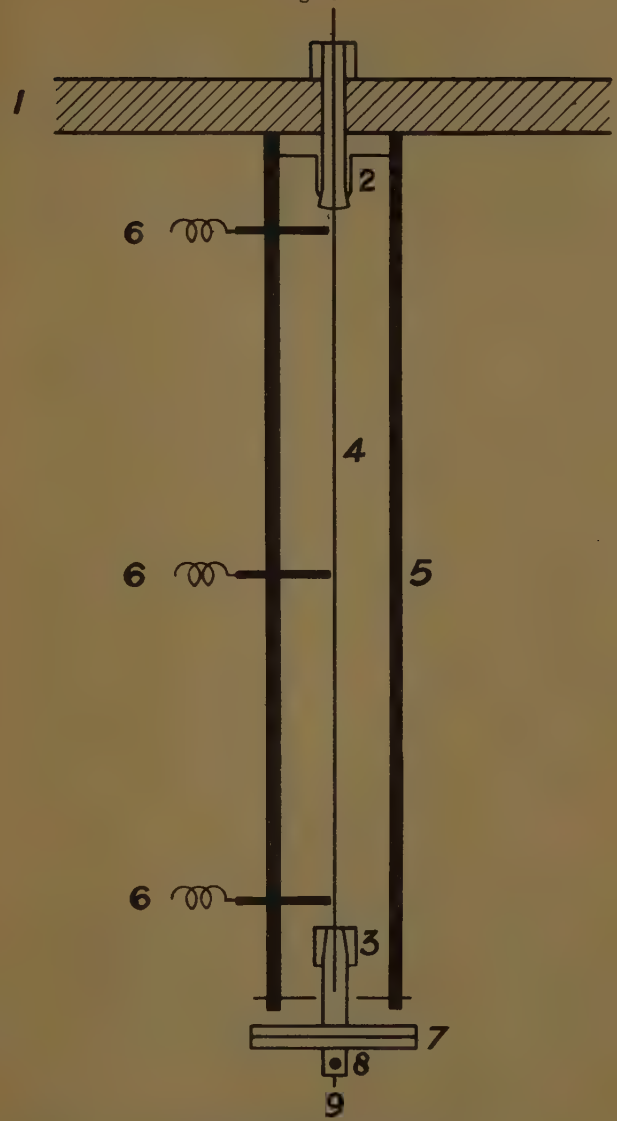
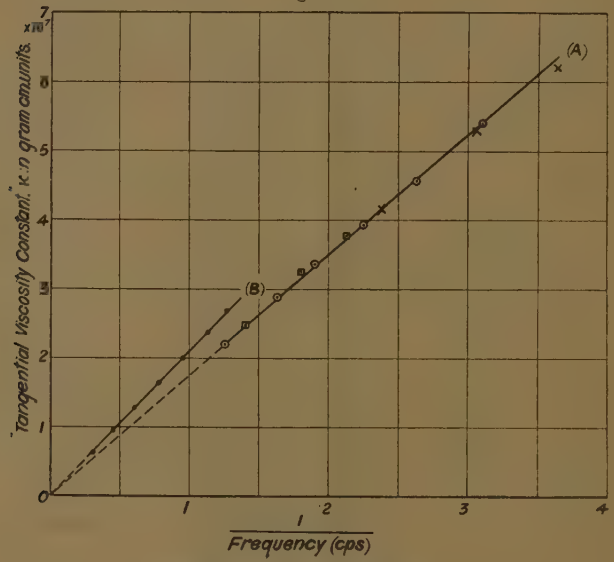


Diagram of torsional apparatus.

1, top supporting plate ; 2 and 3, brass chucks ; 4, specimen ; 5, heating tube ; 6, thermocouples ; 7, load ; 8, mirror ; 9, steadying pin.

and glass. This shows that the modulus of rigidity μ of these materials is independent of the frequency to the accuracy of measurement. The respective values of this modulus at room temperature are collected in Table I. of the Discussion, and call for no comment. The measured values of the damping exponent have been expressed in terms of the apparent value of the "viscosity" coefficient κ of the classic theory, equation (4) being used for the evaluation.

Fig. 2.



Measurements on ebonite at 18° C.

Curve A, 2 mm. diam. rod.
Curve B, 4 mm. diam. rod.

⊙ and • constant load and variable length of shaft.
⊡ and × constant length of shaft and variable load.

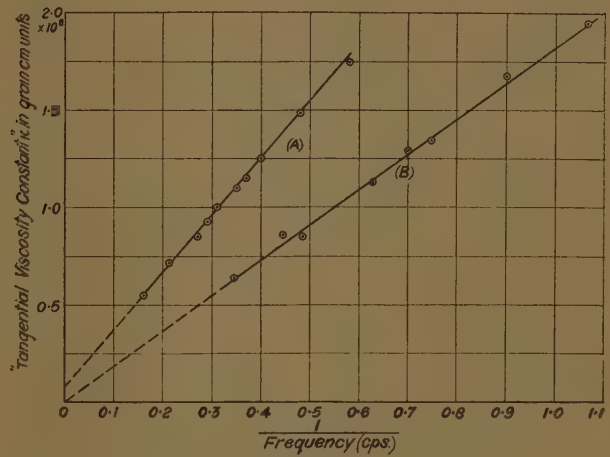
The maximum shear strain generated during the test vibrations was about 1/1800, corresponding, for a steel shaft, to a shear stress of 3 tons/sq. inch.

(a) *Ebonite and Trolitul.*

With rods of ebonite and Trolitul the damping was truly exponential at all amplitudes. However, when κ

was evaluated it was found to be not a constant, but a quantity varying inversely as the frequency. This is shown in figs. 2 and 3; curve A of fig. 2 relates to an ebonite shaft of 2 mm. diameter, and curve B to one of 4 mm. diameter; the derived value of κ at any given frequency is about 20 per cent. greater for the 4 mm. rod than for the 2 mm. one—an effect more likely to be due to a different sample of material than to a true radius effect. Fig. 3 shows the corresponding results for Trolitul; here the points lie perfectly on a straight line which does not pass quite through the origin. The κ

Fig. 3.



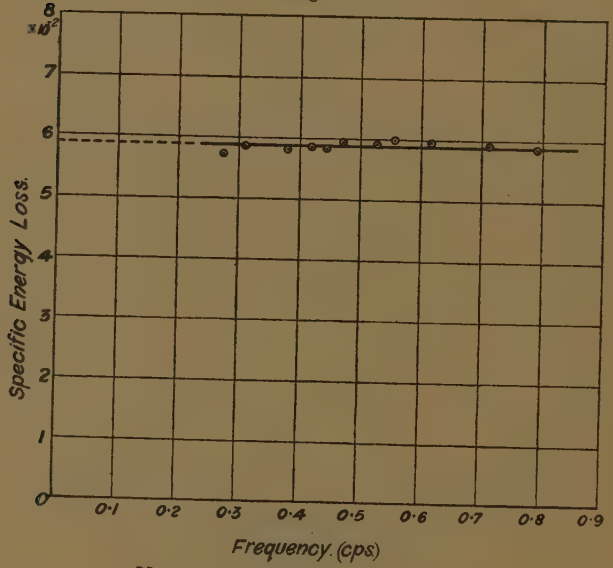
Measurement on Trolitul rod, curve (A) at 16° C.; soft glass rod, curve (B) at 19° C.

values for Trolitul are about twice as large as those for ebonite at the same oscillation frequency.

The significance of the results plotted in figs. 2 and 3 is best appreciated by reference to equation (6); in fig. 4 the specific loss given by this equation, *i e.*, the fractional energy loss per cycle, is plotted against the frequency for the 2 mm. diameter ebonite rod (Curve A, fig. 3). It is seen that, whereas the classic theory leads us to expect the specific loss to increase linearly with frequency, it is actually independent of frequency to the accuracy of measurement.

We must now discuss whether the measured damping exponents are truly attributable to the shaft and not to the supports. There is much evidence in figs. 1 and 2 to suggest that the clamping chucks are not an important source of damping. For to obtain all these observation points referring to frequency by change of length, it was necessary to unscrew the chuck and grip again at a different place : it is most unlikely that fixing conditions producing

Fig. 4.

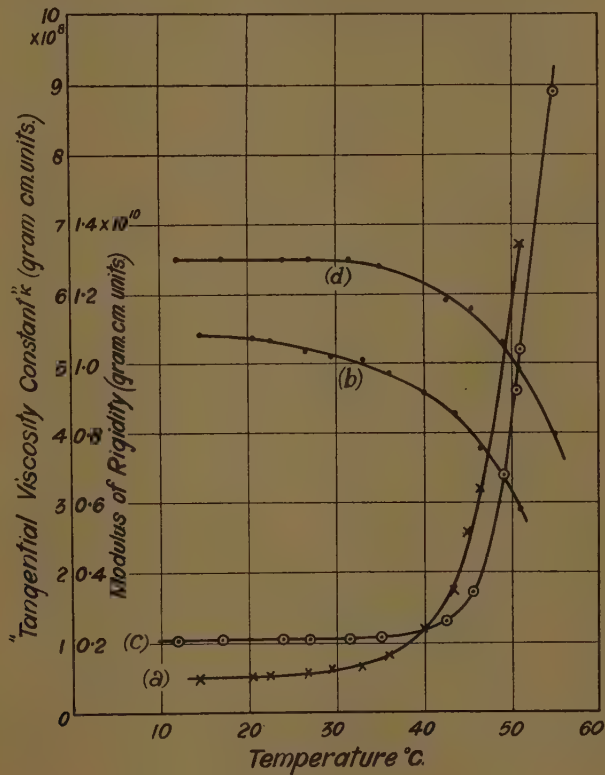


Measurements on ebonite rod at 18° C.

identical damping should always have been obtained. Further, it was found that minor adjustments to the supporting system, such as loosening the fixing screws, did not change the damping by a detectable amount. The damping due to supports and clamps would be a constant, independent of the material of the shafts : yet it was found that the damping factor for steel shafts was only one-tenth that for ebonite. It thus seems certain that the measured damping factors are due dominantly to the shaft material.

The dependence of both κ and μ on temperature are shown in fig. 5. The initial (room temperature) oscillation frequency was 0·365 cycles per second for the ebonite rod and 0·29 cycles per second for the Trolitul : the frequency varied with temperature, and at the limiting

Fig. 5.



Variable temperature measurements on
Ebonite—Curves (*a*) and (*b*) giving κ and μ respectively.
Trolitul—Curves (*c*) and (*d*) giving κ and μ respectively.

observation points, 51° C. and 56° C., with these materials it had fallen to 0·267 and 0·225 cycles per second respectively. A gradual extension of the rods became notice-

able when the temperature exceeded 35° C. for ebonite and 45° C. for Trolitul, and the changes in shaft dimensions have been allowed for in the calculations. For both ebonite and Trolitul there is a continuous decrease of μ and increase of κ with increasing temperature, but in Trolitul the change was too small to measure until the temperature exceeded 30° C.

(b) *Quartz and Glass.*

The damping of one quartz and three glass shafts was measured. One of the latter was a "soft" glass of unknown constitution; another a soda glass (main constituents: SiO_2, 68 per cent.; Na_2O, $18\frac{1}{2}$ per cent.; CaO, $5\frac{1}{2}$ per cent.; Al_2O_3. 4 per cent.; K_2O, 2 per cent.), and the third a lead glass (main constituents: SiO_2, 57 per cent.; PbO, $25\frac{1}{2}$ per cent.; K_2O, $7\frac{3}{4}$ per cent.; CaO, 5 per cent.; Na_2O, $3\frac{1}{2}$ per cent.).

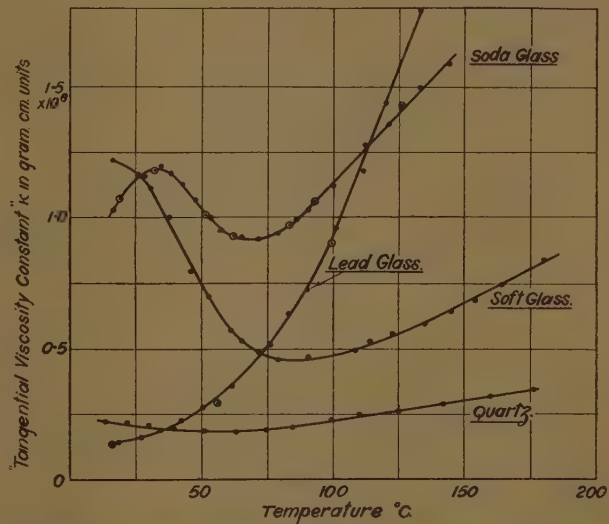
The soft and soda glasses behaved like ebonite and Trolitul in that the amplitude decay was exponential and that the "tangential viscosity constant" varied inversely as the frequency. This is demonstrated in fig. 3 for the soft glass; the observation-points relate to variation of both the shaft-length and the load.

The damping of the quartz and lead glass shafts was very much less than that for the soft and soda glasses, and, moreover, was not truly exponential; the damping exponent showed a continuous increase with increase of amplitude. (This effect occurs also with steel shafts.) In consequence a definite meaning could be attached to the damping exponent only for an oscillation decay from one assigned amplitude to another assigned amplitude. This restriction precluded the possibility of changing the frequency by changing the shaft-length, for this would change the stress produced by a given amplitude. However, three frequencies were obtained by changing the moment of inertia of the load, and for these the quantity κ was found to vary inversely as the frequency, as for the previous materials.

The damping measurements at constant frequency and variable temperature were also made between assigned limits of amplitude. The modulus of rigidity of the quartz and the three glasses increased slightly with temperature, but the increase was only of the order of 2 per cent. at a temperature of 180° C.

The dependence of the apparent viscosity coefficient on temperature is exhibited by fig. 6. For soft glass and soda glass there is both a maximum and a minimum value. The behaviour with temperature seems to be perfectly repeatable. Thus the points marked with a dot in fig. 6 were obtained with a rising temperature, and those marked by a circle with a falling temperature. Also precisely the same curves were obtained for the soda glass when the measurements were repeated after an interval of two days.

Fig. 6.



Variable temperature results on quartz and glass rods at the following frequencies :—

Soft glass, 1·64 ; soda glass, 1·54 ; lead glass, 1·68 ; and quartz, 2·06. Cycles per sec. respectively.

III. Lateral Vibrations.

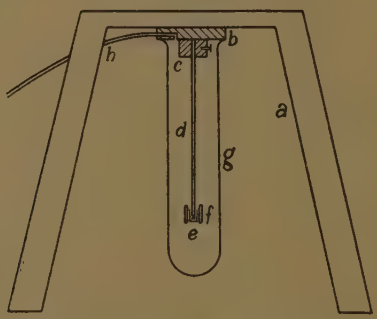
(a) *Apparatus.*

The device consisted of a wooden support *a* (fig. 7) with a brass block *b* screwed on it, which in turn carried a steel clamp *c*, to keep the bar *d* under test fixed at the

top end. A small metal holder *e*, which served as support for weights *f*, was screwed to the free end of the bar.

The vibrations were carried out *in vacuo*, in order to reduce air-damping. In the case of lateral vibrations the air-damping is probably of a turbulent character. As a matter of fact, a viscous resistance of the air would not be eliminated by evacuation, since the viscosity of air is independent of density down to very low values, but a turbulent resistance must be considerably diminished under reduced pressure. A long glass tube *g* with ground flange was fitted to the brass block *b*, a small tube in the block leading to the pump by means of the tubing *h*. The pump was a Metropolitan-Vickers oil pump, fed

Fig. 7.



Apparatus for lateral vibrations.

with Apiezon oil, reducing to pressures of the order of 0·1 mm. mercury.

Vibrations were initiated by means of a strip of iron attached to the holder *e* and a bar magnet outside the glass tube. The frequency of the vibrations up to 2·5 cycles per second was measured by stop-watch, but above this value it was found more reliable to make use of an oscillogram. This was obtained by causing a needle attached to the holder *e* to vibrate over the surface of a rotating drum smeared with vaseline. These measurements were carried out in the open air; the additional damping due to air and needle friction, if not too large, does not affect the frequency. The frequency may be calculated from the wave-length λ of the impression on

the drum-surface by use of the relation $f=\frac{2\pi rn}{\lambda}$, where r=drum radius and n=speed of the drum in r.p.s.

The oscillation decay during the damping measurements *in vacuo* was studied through a microscope directed towards one edge of the lower part of the specimen, and a micrometer scale, each division of which was 0·0074 cm., and the damping factor Δ calculated as before from the time taken for the amplitude to decay from one particular value to another.

The frequency was varied by changing the length of the specimen and also the weight attached to the free end. The practical range of frequency was limited, however, at both the lower and higher ends. At the higher frequencies the decay occurred so rapidly as to render exact visual measurements impossible, while at low frequencies the theory of the motion of the system is complicated by the increasing significance of gravitational forces. As indicated below, the method is only useful so long as the gravitational force does not become the dominating controlling influence. For these reasons the frequency range was restricted to 1–10 c.p.s. approximately.

(b) *Theory.*

The theory of lateral vibrations of loaded bars is known *, so we give here only the derivation of an approximate formula, sufficient for our purpose, based on Rayleigh's method of approximation with regard to the mass of the bar itself. The salient point of this method is to assume that for sufficiently slow vibrations (dimension of the bar small against the wave-length of the fundamental) the deflexion curve is that of the static case.

The bar has a length l (fig. 8), thickness a, and width b; it is fixed at $x=0$, and $x=l$ represents the free end supporting the added mass M. Motion takes place in the plane of the paper, in the direction y. The deflexion curve is given by

$$y=\frac{P}{6EI}(3lx^2-x^3),$$

* S. Timoshenko, ' Vibration Problems in Engineering,' London, 1928, p. 221.

where P=force acting horizontally at the end producing the static deflexion, E=Young's modulus of elasticity, and $I=\frac{ba^3}{12}$.

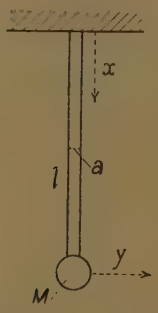
It is further assumed that the motion is a simple harmonic one; thus we have for the deflexion at $x=l$

$$y_l=Ae^{-\Delta t}\sin\omega t,$$

where A=initial amplitude, Δ=damping factor, and $\omega=2\pi f$ (f=frequency). From this it follows for the velocity

$$\dot{y}_l=\omega y_l.$$

Fig. 8.



Section across bar and weight.

In order to write down the differential equation, we require the mass correction owing to the mass M_0 of the bar itself. The kinetic energy is equal to

$$\tfrac{1}{2}(My^2+M_0\bar{y}^2)\omega^2,$$

where $\bar{y}^2$ is the mean square value, calculable by integrating the static expression for y^2. We obtain $\bar{y}^2=\frac{33}{140}y_l^2$, the equivalent mass being $\left(M+\frac{33}{140}M_0\right)$. The acceleration term, considered as rotation with regard to the origin ($x=0$), is, therefore,

$$\left(M+\frac{33}{140}M_0\right)l\ddot{y}_l.$$

The restoring couple owing to elasticity is Pl, i. e., $\frac{3\mathrm{EI}y_l}{l^2}$.

The restoring couple due to gravity is $g(\mathrm{M}y_l+\mathrm{M}_0\bar{y})$, where the average value y is again calculable by integration, being equal to $(3/8)y_l$.

As to the frictional term, we write it as proportional to velocity and a factor ξ, the so-called "longitudinal viscosity," the couple being

$$\frac{3\xi\mathrm{I}}{l^2}\dot{y}_l.$$

The differential equation is, therefore,

$$\left(\mathrm{M}+\frac{33}{140}\mathrm{M}_0\right)\ddot{y}_l+\frac{3\xi\mathrm{I}}{l^3}\dot{y}_l+\left[\frac{3\mathrm{EI}}{l^3}+\frac{\left(\mathrm{M}+\frac{3}{8}\mathrm{M}_0\right)g}{l}\right]y_l=0.$$

Thus

$$\omega^2=\frac{3\mathrm{EI}}{l^3\left(\mathrm{M}+\frac{33}{140}\mathrm{M}_0\right)}+\frac{\mathrm{M}+\frac{3}{8}\mathrm{M}_0}{\mathrm{M}+\frac{33}{140}\mathrm{M}_0}\cdot\frac{g}{l} \quad . \quad . \quad (7)$$

and

$$\Delta=\frac{1}{2}\frac{\xi}{\mathrm{E}}\left(\omega^2-\frac{\mathrm{M}+\frac{3}{8}\mathrm{M}_0}{\mathrm{M}+\frac{33}{140}\mathrm{M}_0}\cdot\frac{g}{l}\right). \quad . \quad . \quad . \quad (8)$$

The terms containing M_0 are only approximate and have only the character of correction terms, since generally $\mathrm{M}>\mathrm{M}_0$.

Equation (7) shows that if the second term of the right-hand side predominates, *i. e.*, if (neglecting M_0) the condition

$$\mathrm{M}l^2<\frac{3\mathrm{EI}}{g}$$

is not fulfilled, the whole motion is controlled by gravity mainly, the device approaching more a pendulum with $\omega^2=g/l$, in which case (7) becomes unsuitable for the evaluation of E, and similarly (8) for the evaluation of ξ.

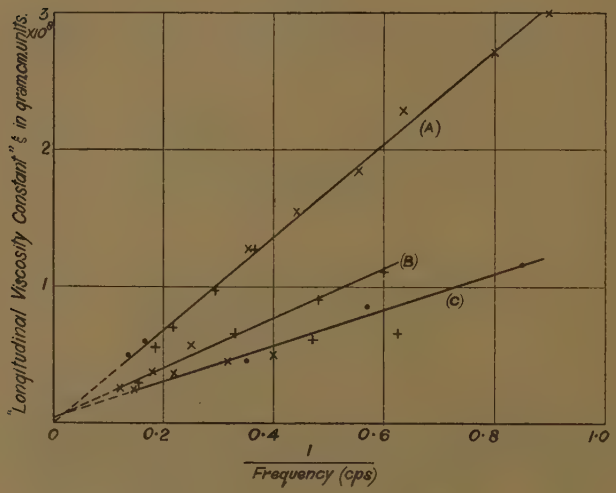
In the measurements the following procedure was adopted. Both weight and length were varied, but only

so long as equation (7) gave a fairly constant value for the modulus E, which could be calculated from the frequency. A deviation from the constant value of E, given by means of a static measurement, indicated that the lower frequency limit was reached and the equation no longer reliable. In the range for which this test was satisfied, ξ was calculated by means of equation (8).

(c) *Experimental Results.*

The experimental results obtained on the dielectric materials, glass, walnut wood, and ebonite, are shown

Fig. 9.

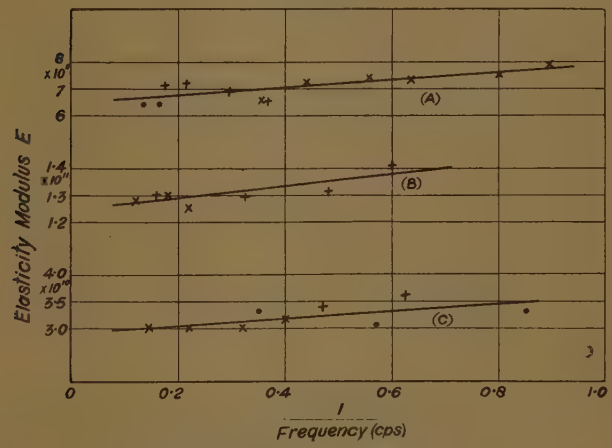


Measurements at room temperature.
Curve (A), glass ; curve (B), wood ; curve (C), ebonite.

in figs. 9 and 10. The specimens had a width of about 1·5 cm., a thickness of 0·15 to 0·30 cm., and a length of from 20 to 60 cm., while the weights attached to the free end varied between 5 and 270 grams. In each of these figures each length is characterized by a different mark, the variation of frequency within the range available with each particular length being effected by varying the weight.

In keeping with the result given by the torsional vibration measurements, fig. 9 shows that the "viscosity" coefficient ξ is not a constant, but varies approximately inversely proportional to the frequency of lateral vibration. That the individual experimental points are probably somewhat in error (due, presumably, to the approximate character of equations (7) and (8)) is revealed in fig. 10 by the apparent 10 per cent. variation of the elasticity modulus E over the frequency range considered, but this does not affect the general nature of the above conclusion, since the variation of ξ within this range of

Fig. 10.



Measurements at room temperature.
Curve (A), glass; curve (B), wood; curve (C) ebonite.

frequency is of the order of 500 per cent. The results may therefore be expressed approximately by

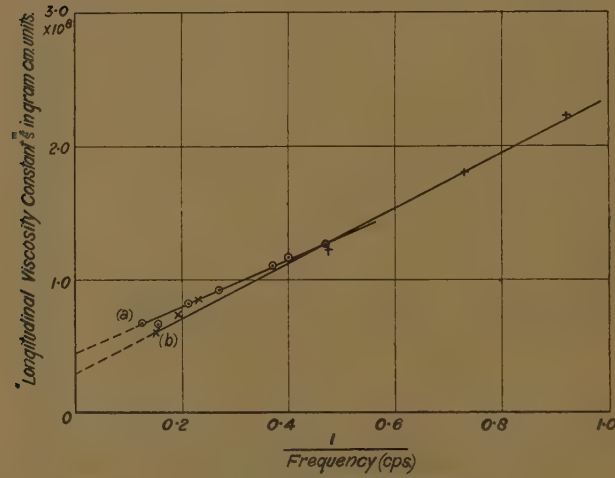
$$\xi f = \text{constant}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

In addition, two curves on copper and steel are given (fig. 11). Contrary to the observations on torsional oscillations, the decrement was found to be fairly constant with decaying amplitude. At least no pronounced decrease of the decrement with decreasing amplitude was noticeable. The curves relating ξ with frequency show a behaviour

very similar to the dielectric materials, with the exception that when extrapolated they do not approach the origin so closely.

It is difficult to decide whether this departure from the more simple behaviour is real, or eventually caused by a loss in the support. In order to throw some light on the question, the metal clamp *c* (fig. 7) was replaced by an ebonite clamp of exactly the same shape. A few measurements on metal bars showed a very small but measurable increase of ξ of some 7 per cent., thus indicating

Fig. 11.



Measurements at room temperature.

Curve (*a*), steel specimen, ⊙.
Curve (*b*), copper specimen, × +.

that departure from eq. (9) might be accounted for by a certain decrement of amplitude owing to loss in the support.

Since, in spite of this uncertainty, the general behaviour of the ξ–f curve is similar to those obtained from dielectrics, it was thought worth while to include them here, thus indicating that the quantitative law governing loss is probably very similar with both groups of materials.

IV. Theoretical Conclusions.

(a) *Measurements at Constant Temperature—Effect of Frequency.*

The conclusion to be drawn from both the torsional and lateral vibration measurements is that the so-called "viscosity constant" of the classical theory is not a constant but is closely inversely proportional to the frequency. The experimental result that κf and ξf for the respective cases are sensibly constant is identical with the statement that the fractional loss of energy per cycle is constant. This follows from equation (6), and from the analogous equation which results for the lateral vibrations on neglecting the correction term of equation (8).

The following table gives the values of this specific loss at room temperature for the materials covered by the paper; the values relating to torsion and bending of the same type of material are not the same, though this is to be expected.

Table I.

Specific Loss, calculated from Experimental Data.

Material.	Torsional measurements.			Lateral vibration measurements.		
	μ.	κf.	$\delta = 4\pi^2 \frac{\kappa}{\mu} . f$.	E.	ξf.	$\delta = 4\pi^2 \frac{\xi}{E} . f$.
Ebonite .	$1{\cdot}18 \times 10^{10}$	$1{\cdot}75 \times 10^{7}$	$58{\cdot}7 \times 10^{-3}$	$3{\cdot}1 \times 10^{10}$	$1{\cdot}3 \times 10^{8}$	170×10^{-3}
Trolitul .	$1{\cdot}3 \times 10^{10}$	$3{\cdot}1 \times 10^{7}$	$95{\cdot}7 \times 10^{-3}$			
Quartz . .	$3{\cdot}48 \times 10^{11}$	$4{\cdot}6 \times 10^{7}$	$5{\cdot}2 \times 10^{-3}$			
Glass :						
Soft . .	$2{\cdot}56 \times 10^{11}$	$1{\cdot}80 \times 10^{8}$	$28{\cdot}3 \times 10^{-3}$	$7{\cdot}0 \times 10^{11}$	$3{\cdot}4 \times 10^{8}$	19×10^{-3}
Soda . .	$2{\cdot}65 \times 10^{11}$	$1{\cdot}60 \times 10^{8}$	$24{\cdot}0 \times 10^{-3}$			
Lead . .	$2{\cdot}30 \times 10^{11}$	$2{\cdot}40 \times 10^{7}$	$4{\cdot}2 \times 10^{-3}$			
Wood . .	. .	. .	. .	$1{\cdot}3 \times 10^{11}$	$1{\cdot}75 \times 10^{8}$	54×10^{-3}
Copper .	. .	. .	. .	$1{\cdot}25 \times 10^{12}$	$2{\cdot}0 \times 10^{8}$	$6{\cdot}4 \times 10^{-3}$
Steel . . .	. .	. .	. .	$2{\cdot}0 \times 10^{12}$	$1{\cdot}7 \times 10^{8}$	$3{\cdot}4 \times 10^{-3}$

The above observation conforms to that of Kimball ('Vibration Prevention in Engineering,' New York, 1932, p. 133), derived from measurements by an entirely different experimental procedure, namely, the lateral deviation of an overhanging rotating rod. His results are given in terms of the logarithmic decrement, which

is exactly one-half the specific loss per cycle; when converted to the latter they give the following values :— Glass, 13×10^{-3}; wood, 44×10^{-3}; nickel steel, $4{\cdot}6 \times 10^{-3}$. It is seen that these figures, and those given in the above table correspond reasonably closely.

Our results can be expressed, following the terminology of alternating current theory, by the statement that the loss angle, α, is independent of the frequency, since this angle is given by the relation

$$\tan \alpha = \frac{\kappa\omega}{\mu} = 2\pi \frac{\kappa f}{\mu},$$

in which κf and μ are constants for a given material.

As mentioned in the introduction, an explanation of this behaviour may be sought in the possibility either that the "viscosity constant" of the classical theory is not actually a constant, but some function of the frequency of vibration, or that the damping term is not dependent on the velocity, but is a function of the amplitude of the vibration. What is clear is that an interpretation of damping on the classical viscosity basis is quite inappropriate—it is only deference to tradition which has led us to present the results in terms of this analysis.

1. *Solid Friction.*

Let us consider first the second of these two possibilities, which Kimball discusses under the heading of "Solid Friction." He has shown that the amplitude decay is exponential and the specific loss independent of frequency if the damping term of the fundamental differential equation (compare eq. (1)) is expressed as the product of a constant and the first power of the amplitude of vibration. The cases in which the decay factor Δ is not constant, but is dependent on the amplitude of vibration, can then be accounted for by assuming the frictional force to be a different function of this amplitude. Thus, where the decrement decreases with decreasing amplitude, as with quartz and lead glass, the force must be proportional to a power of the amplitude in excess of unity.

The important feature of the motion is the constancy of the fractional energy loss per cycle. This is clearly

consistent with the existence of a tiny, frequency-independent, hysteresis loop. For an exponentially decaying sinusoidal vibration this loop is an ellipse of constant shape, for which the residual strain at zero stress is δ/π of the maximum strain. The values of the specific loss δ given in the preceding table for the materials tested suggest that this residual strain should lie between about 3 per cent. for Trolitul and 0·1 per cent. for glass, and it is doubtful whether such a thin loop could be detected in static measurements. It is possible that the departure from exponential damping observed with lead glass, quartz, and some metals is due to the shape of the hysteresis loop being a function of the maximum strain, an effect quite familiar in magnetism. Our data refer to the range well within what is referred to normally as the limit of proportionality between stress and strain; static tests usually extend beyond the yield-point, the permanent set increasing, according to Rowett *, with the third power of the maximum stress.

It is not possible to enlarge on the physical processes involved. Loss mechanisms concerned with the slip of crystallographic surfaces are referred to the yield range †, whereas only tentative theories exist for the loss in the proportional range where it is supposed to be of thermodynamic origin ‡.

2. *Complex Viscosity.*

The other alternative is to consider the behaviour of elasto-viscous materials and regard solids as an extreme member of this group. At least for amorphous, plastic materials this conception seems to be promising, as shown recently by one of the authors §.

In such materials the main features of a liquid and a solid are considered as coexistent, an assumption much strengthened by recent investigations on the quasi-crystalline structure of liquids ||. In its simplest form

* Rowett, Proc. Roy. Soc. vol. lxxxix. p. 528 (1913).

† G. I. Taylor, Fourth International Conference of Applied Mechanics, 1934, p. 113.

‡ S. Timoshenko, 'Strength of Materials,' ii. p. 675 (1931).

§ A. Gemant, Trans. Farad. Soc. xxxi. p. 1582 (1935); 'Physics,' vi. p. 363 (1935).

|| W. E. Morrell and J. H. Hildebrand, J. Chem. Phys. iv. p. 224 (1936).

a crystal is characterized by a finite resilience in series with a resistance, whereas a liquid by a resistance term alone. Denoting the total resistance, called the complex viscosity, by V, we have (for the torsional case)

$$V=\kappa-i\frac{\mu}{\omega}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

where κ and μ are the resultant resistance and modulus. As to the former, its value is given by

$$\kappa=\eta \cdot \frac{1+(\kappa_0/\mu_0)\tau_0\omega^2}{1+(\tau_0\omega)^2}, \quad . \quad . \quad . \quad . \quad (11)$$

where κ_0 and μ_0 are the crystal coefficients, η the liquid one, and $\tau_0=\eta/\mu_0$.

Qualitatively (11) is in agreement with our results, since κ decreases with increasing frequency. The temperature variation is also in accord with (11), since in the solid range κ is expected to rise steadily with temperature, which (apart from an initial maximum with glass) is the case in our experiments (see next section). Similar results have been obtained by Jida * for quartz.

However, it is not possible to reproduce the empirical relation (9) quantitatively by (11). Similar discrepancies have been observed even for typical elasto-viscous colloidal solutions, so a complete agreement with solids cannot by any means be expected. Equation (11) is only the simplest possible form of the real part of a complex viscosity. Agreement with experiments concerning both alternating and relaxation processes on elasto-viscous media can be obtained by changing the exponent of ω in (11) to some lower figure near unity †. A physical explanation of this change of the exponent may be, for instance, a superposition of several time-constants τ.

This conception, though unlikely to be generally applicable, may be of importance in the case of materials with finite plasticity, as, for example, for materials strained beyond the yield-point or operated at high temperatures. For normal operating conditions the discussion of the preceding section on solid friction seems more appropriate.

* K. Jida, Tokyo Univ. Earthquake Research Inst. Bulletin, xiii. p. 665 (1935).

† A. Gemant, ' Physics,' vii. p. 311 (1936).

(b) *Measurements at Variable Temperature.*

In the light of the above discussion it appears possible that the behaviour of ebonite and Trolitul (fig. 5) can be explained by attributing the first horizontal branch to a sort of solid friction, and the sudden rise at higher temperatures to the influence of a liquid viscosity η. The well-known exponential decrease of η with temperature must show itself in an increase of κ according to equation (11), since κ is the equivalent *series* resistance, whereas η is considered as in parallel with the elasticity.

No definite explanation of the maxima which occur in the curves of fig. 6 for two of the glasses tested can be put forward. The effect is suggestive of the existence of more than one mechanism, but its clarification will require an extended investigation covering a wide frequency as well as temperature range carried out on materials of systematically varied constitution.

V. Summary.

The methods of free torsional and free lateral vibrations have been employed to study the internal damping of several solid dielectric materials—ebonite, trolitul, quartz, glass and wood—over the frequency range 0·3 to 10 cycles per second. The classical assumption that the friction is of a viscous character is found to be an erroneous one. The "viscosity constant" of this classic analysis is found to vary inversely as the frequency to a first approximation. This corresponds to a constant energy loss per cycle. This behaviour is in agreement with certain observations of A. C. Kimball, and is probably to be explained in terms of solid friction. An alternative explanation based on the conception of a complex viscosity is also considered. The dependence of the elasticity modulus and the damping on temperature has been studied between 20 and 180° C. The general tendency is for the former quantity to decrease and the latter to increase with temperature, although the damping-temperature curve for two glass samples exhibits a low-temperature maximum preceding this normal behaviour.

The authors are indebted to Mr. E. B. Moullin for much helpful suggestion and discussion, and to Messrs. Metropolitan-Vickers Electrical Co., Ltd., and the Department of Scientific and Industrial Research for two grants enabling the research to be performed.

XCIV. *Lattice Dimensions of Electroplated and Normal Chromium.* *By* W. A. Wood, *M.Sc., Physics Department, National Physical Laboratory, Teddington, Middlesex* *.

[Plate VIII.]

THE X-ray diffraction spectrum of bright electroplated chromium is characterized by abnormally diffuse lines ; the diffusion is such as to prevent accurate determination of the positions of the lines, and therefore of the lattice spacings, by the customary methods. The purpose of the present work is a comparison of the position of corresponding spectral lines in electroplated and in normal pure chromium by means of a technique which overcomes the difficulties associated with the diffusion of the lines.

The method consists, essentially, of superposing on a diffraction ring from electroplated chromium a second ring from pure chromium which has been annealed sufficiently to produce large and perfect grains : the electroplate gives, as indicated above, a diffuse halo : the pure chromium, owing to the large grains, gives instead a series of isolated sharp spots spaced at intervals upon the halo. Microphotometer records are taken radially across the composite ring at the points where a diffraction spot is superposed on the halo ; each record thus shows a sharp peak, due to the spot jutting out from the extended trace of the diffuse halo. A comparison of the positions of the peak and the maximum of the underlying trace then provides a sensitive indication of any relative shift, and therefore of any change in lattice spacings. The sensitivity is greater than that which could be obtained by the usual " powder " method. The use of the annealed large-grained metal as a standard ensures that the grains are free from distortion, such as that unavoidably introduced by filing or working the metal in order to get the smaller grain which gives the continuous " powder " type of diffraction line. Such lines are seldom so sharp, for instance, as those from an ionic salt, and cannot possibly lend themselves to the accuracy of measurement which is commonly claimed for them [1]. The reflexions from a well-annealed grain,

* Communicated by Dr. G. W. C. Kaye.

however, are often of the same order of sharpness as those from calcite.

The work forms part of researches on the crystalline structure of metals in an abnormally hard state. Previous results indicate that this condition gives rise to a diffuse type of spectra in which the degree of hardening is proportional to the line-broadening. Chromium plate is an outstanding example of this effect: it has been found that the size of the crystallites, deduced from the broadening of the lines, is of the order of 10^{-6} to 10^{-7} cm., values much less than those to which the grains of a pure metal could be reduced by cold-working [2], and it has also been suggested that the main function of the foreign atoms, co-deposited during electroplating, is the inhibition of the growth of the grains to the normal order of size [3]. The present work measures the effect of the co-deposited gases on the dimensions of the crystal lattice with a view to locating the gas. An earlier preliminary test, based on the usual methods, was not sufficiently sensitive to show any change [2].

Composite X-ray Photographs.

Figs. 1 and 2 (Pl. VIII.) illustrate the types of composite photographs used; fig. 1 shows the diffuse (211) diffraction halo from bright chromium plate, together with the superposed diffraction spots from the (211) planes of the pure chromium standard; fig. 2 shows *exactly the same* individual diffraction spots as fig. 1, but they are superposed, in this case, on the diffraction ring from the (211) planes of annealed chromium. This ring possesses a sharp doublet formation owing to the growth in size and perfection of the grains as a result of the heat-treatment used for normalizing; the outer ring is the reflexion of the $K\alpha_1$ component and the inner that of the $K\alpha_2$ component of the CrK wave-length. This photograph is a means of identifying the diffraction spots in fig. 1; it is immediately possible to see from a comparison of figs. 1 and 2 those spots in fig. 1 which are formed, for instance, by reflexion of the α_1 wave-length, since these lie on the outer diffraction ring of fig. 2. The method turns on the reproduction of exactly the same diffraction spots each time a composite photograph is made, since it is possible by so doing to allow automatically for variations in divergence and intensity distribution in the incident beam,

which, in the usual methods, lead to shifts in the apparent centre of a line. This reproducibility was ensured by a jig, embodied in the specimen holder of the apparatus used, which caused the incident X-ray beam always to fall on the same point of the specimen. Alternatively, the photographs could be taken in the following order : (*a*) chromium plate, (*b*) pure standard (for fig. 1), and then, changing the film in position, but leaving the standard specimen untouched. (*c*) pure standard, and (*d*) normalized chromium or other specimen for comparison (for fig. 2). The photographs which included annealed electro-deposited chromium were used also to find the effect of the heat-treatment on the lattice dimensions.

It is at once evident from these figures that the broad halo from the bright electro-deposited material has not diffused in a symmetrical way about the ring of diffraction spots indicated by the pure standard. This is seen most easily by consideration of the α_2 spots, recognizable by reference to the inner ring of the doublet from normalized chromium. Particular α_2 spots are indicated by the arrows drawn in the figures ; these spots appear actually to lie on the inner edge of the diffuse halo instead of somewhere towards the middle part, as they would if the halo had been produced from fig. 2 by as much diffusion inwards as outwards from the centre. There has been, therefore, a general increase in the mean radius of the halo in comparison with the radius of the ring from the annealed chromium or with that of the ring forming the locus of the diffraction spots from pure chromium. The rings are formed by deviation of the incident beam backwards through angles greater than 90°. An increase in the radius of a ring corresponds to a decrease in the angle of deviation, and therefore, by the Bragg law, to an increase in the dimensions of the crystal lattice. The spacings of the lattice of electro-deposited chromium are therefore greater than those of pure or of normalized chromium. Quantitative measurements were obtained from the microphotometer records.

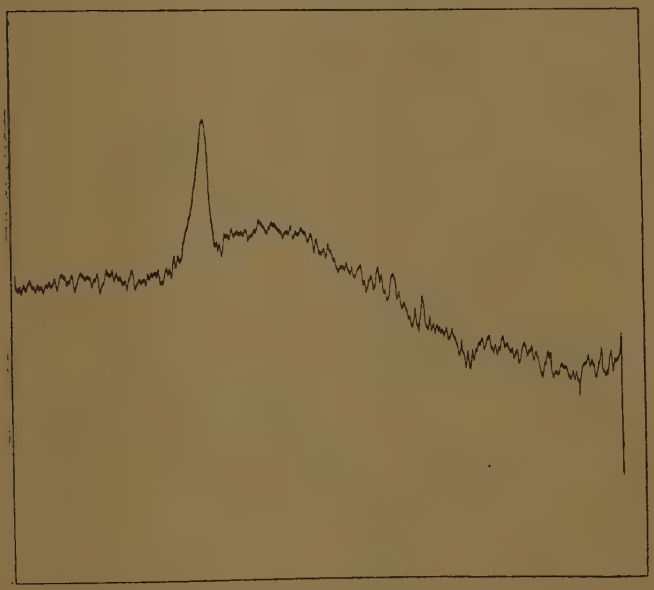
Microphotometer Records of Composite Spectra.

The general shape of the diffuse halo is shown by fig. 4 *c*, obtained by traversing the halo at a point free from diffraction spots.

The position of an α_2 diffraction spot on the halo is indicated by the sharp peak in fig. 3. It is clear that the peak juts out just when the trace of the halo is starting to form, and that there is no appreciable broadening of the halo to the left of the peak : the latter, therefore, falls almost on the inner edge of the halo.

The position of an α_1 diffraction spot relative to the halo is shown by fig. 4 *a*. It can be shown that if the

Fig. 3.



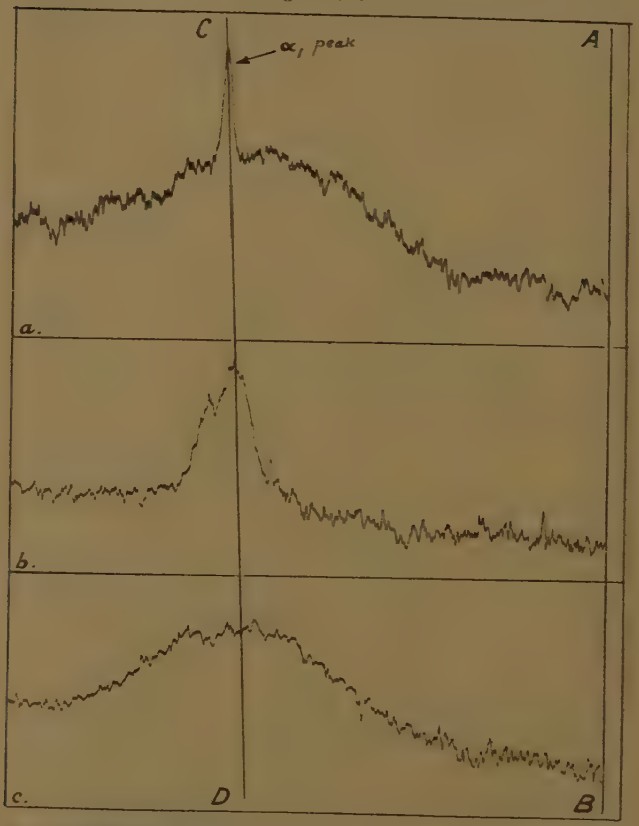
Showing standard of α_2 peak relative to broad line from bright chromium plate.

spacings of the pure and electro-deposited chromium were the same, the α_1 diffraction spots would coincide with the maximum of the halo. Fig. 4 *a* shows that this is not the case ; the α_1 spot is distinctly to the left of the maximum.

It is possible to demonstrate the shift by the usual mode of measurement although, owing to possible changes in the films and records during processing, the method is

less reliable for quantitative results. The distance between the rings and the edge of the photograph is traversed by the microphotometer: this edge is the circular boundary seen in figs. 1 and 2 (Pl. VIII.), and is sufficiently sharp to

Figs. 4 *a*, *b*, *c*.



Slope of CD shows shift of spectra of (*a*) standard, (*b*) annealed, and (*c*) plated chromium relative to zero line AB.

cause an abrupt end in the trace of the microphotometer record. In fig. 4 these ends are mounted vertically, forming the line AB, for the three records: (i.) from the composite

photograph including an α_1 diffraction spot, (ii.) the doublet from normalized chromium, and (iii.) the halo from the electroplate. The variation in the distance from AB of the α_1 peak, of the α_1 component of the normalized doublet, and of the maximum of the halo is demonstrated by the slope of the line CD joining these points.

Results.

The pure chromium used as standard was prepared by Dr. Adcock, Metallurgy Department, National Physical Laboratory [4].

The chromium deposits were prepared in the normal type of plating bath, consisting of 25 per cent. CrO_3 ("Analar" purity) and sulphuric acid content corresponding to 0·05 N. The bright type of deposit was obtained by working the bath at about 40° and the dull type at room temperature. The annealing was done at 700° *in vacuo*, the degree of annealing being fixed by the time. The deposits were on copper, and were ground to have a flat surface for examination.

The X-ray spectrograph used was an apparatus specially designed for accurate work. The special features included the arrangements for ensuring that successive specimens could be placed in exactly the same position in the holder, and that the surface of the photographic film should be held flat by clamping at the centre and at the edges. The films were quarter-plate size, and could be interchanged without disturbing the rest of the apparatus. Also care was taken, by using a long and narrow tubular slit, that the reflexions from the large grains of the standard should not be unduly displaced from the theoretical positions on account of divergence of the incident beam. Whilst this point is not essential for comparison work, since this was based on the use of exactly the same reflexion spots from the standard, it comes in when absolute measurements are made. These are given below.

(a) *Spacing of Pure Chromium.*—For the absolute spacing the diffraction spots due to the pure chromium were obtained, and the radius of the ring on which the α_1 spots lay found by means of a travelling microscope. The main difficulty in precision determinations of spacing is the location of the surface of the film and allowance for the thickness of the emulsion; this was minimized

by using a process film, with thin emulsion on one side only, and clamping the film in a holder so that the emulsion was pressed against a specially lapped flat plate of metal, the surface of which was fixed at a known distance from the flat specimen by a spacing gauge; the microscope objective used for measurements on the diffraction spots was then chosen to have a very small depth of field, and was focussed on the surface of the emulsion. The diffracted X-rays passed through the celluloid base before striking the emulsion of the film.

The measurements gave for the edge of the cubic unit cell $a = 2{\cdot}879_6$ Å. at 18° C.

(b) *Spacing of Electroplated Chromium.*—The distance of the α_1 diffraction peaks of the pure standard from the maximum of the halo on which they lay was averaged from microphotometer records of the type 4 (*a*): the microphotometer itself introduced a magnification of seven times. From these measurements, taking the value of the cube side of the pure chromium to be that found above, the fractional increase in the spacing of the normalized chromium was

$$\frac{\delta a}{a} = +0{\cdot}00093\,;$$

whilst the increase in spacing of the bright type of electroplated chromium over the pure standard was

$$\frac{\delta a}{a} = +0{\cdot}00205.$$

It was found that the spacings of various samples of the bright electroplate varied very little amongst themselves. Such specimens, as shown by fig. 1 (Pl. VIII.), give extremely broad spectral lines. The constancy of the spacings therefore suggests that the lattice has taken up the maximum amount of co-deposited gas when the chromium is in the bright state.

The lattice spacing of the dull matt type of chromium plate was found invariably to be less than that of the bright form, but varied from specimen to specimen. The values averaged at an increase in spacing given by

$$\frac{\delta a}{a} = +0{\cdot}0015,$$

being intermediate between that of the annealed and the pure chromium. The dull chromium gives spectral lines

which are less broad than those given by the bright type, and therefore possesses a somewhat larger size of constitutent crystallite. This suggests that the larger size is permissible as a result of a less gas content and associated distortion.

The expansion of the unit cell of the deposited material of either type is very small, but proves that a proportion of the gases liberated during the standard methods of plating are condensed actually within the lattice, as distinct from any which may be adsorbed on the surfaces of the constituent crystallites. The fact that, by ordinary annealing, it is impossible to reduce the lattice spacing to normality, although large quantities of gas are driven off, indicates that the gas within the lattice is the most difficult to remove. It is concluded therefore that the greater part of the gases co-deposited with the metal, and responsible for the characteristic properties of chromium plate, are adsorbed on the surface of the crystallites (10^{-6} to 10^{-7} cm.), in view of the large emission which could be obtained without appreciable change in lattice dimensions and that the gas actually within the lattice, though producing a definite effect, is comparatively small. The lattice changes are much less than, for instance, those associated with the absorption of hydrogen by palladium.

Summary.

A comparison is made of the lattice spacings of pure, electro-deposited, and annealed chromium, employing a technique developed to study the diffuse type of spectral line characteristic of chromium plate. It is shown that the unit cube of the electro-deposited material is greater than that of the pure chromium, and that the expansion persists after ordinary annealing treatment. Quantitative measurements are made of the lattice spacings of the pure and electro-deposited forms. It is concluded that the greater part of the gases co-deposited in chromium plating and responsible for its particular properties is adsorbed on the constituent crystallites.

References.

(1) Compare ' Crystal Structure,' W. P. Davey, p. 167 (McGraw Hill, 1936).
(2) W. A. Wood, Phil. Mag. xii. p. 853 (1931).
(3) W. A. Wood, Trans. Far. Soc. xxxi. p. 1248 (1935).
(4) F. Adcock, Jour. Iron & Steel Inst. (May 1927).

XCV. *On the Response of a Vane Anemometer to an Air-stream of Pulsating Speed.*

By E. Ower, *B.Sc., A.C.G.I., of the Aerodynamics Department, National Physical Laboratory* *.

[Plate IX.]

Summary.

The behaviour of a vane anemometer in a pulsating air-stream is examined theoretically. It is shown that the anemometer over-estimates the average air-speed, and an expression is given for the magnitude of the error in an air-current whose speed pulsations follow a sine law. In order that the equation of motion may be integrable analytically a small term is neglected, and this step is shown to be justified by comparing the approximate solution with the results of numerical integration of the complete equation of motion.

The response of a particular vane anemometer to a pulsating air-stream has been observed in the wind tunnel, and the results of the experiments confirm the theoretical conclusions.

1. *Introduction.*

In a discussion of the theory of the vane anemometer published some time ago [1] the writer showed that in a current of air of pulsating speed the rotor would rotate at a higher average rate than that corresponding to a steady current of speed equal to the time-average of the pulsations. This conclusion was in agreement with the general experience of users of these instruments and with the results of a theoretical and experimental investigation by Rateau [2], but it conflicted with a later analysis by Glauert [3], who deduced that with a properly designed anemometer there is no appreciable over-estimation.

The writer's interest in the problem was revived by a recent enquiry from some users of vane anemometers, who criticized a certain assumption that had been made in the earlier analysis in order to make possible the integration of the equation of motion. The theory has now been re-examined; the assumption in question

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has been justified by numerical integration of the complete equation; and an experimental investigation has shown that a particular anemometer did, in fact, over-estimate by an amount that agreed with the theoretical prediction within the limits of the experimental accuracy.

I. Theory.

2. The equation of motion of the vanes takes the usual form

$$2\pi I\frac{dn}{dt}-Q+F=0, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where

n is the instantaneous rate of rotation of the rotor at time t,

I is the moment of inertia of the rotor about the axis of rotation,

Q is the driving torque due to the wind, and

F is the frictional torque at the bearings.

Values of Q and F are obtained from the calibration curve of the anemometer in a steady air-current by the methods described in ref. [1], and from a knowledge of the lift and drag coefficients of the flat plates forming the vanes. Values of these coefficients were obtained by Dr. W. J. Duncan and the writer in an experimental investigation [4] made to check the values for flat plates given by Eiffel [5].

The calibration curve of an anemometer is, in effect, a relation between the true air-speed V and the rotational speed n under steady conditions, *i. e.*, when $Q = F$. Except at very low speeds, this calibration curve is always found to be a straight line, and, in the case of a typical anemometer, calculation of Q and F by the method outlined above showed that over the linear portion of the calibration curve Q and F could be represented by the equations

$$Q/\rho D^3V^2=a+b\frac{nD}{V} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and

$$F = KV, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where

a, b, and K are constants for a given instrument,

D is the diameter of the rotor, and

ρ is the air-density.

By substituting in (1), the equation of motion becomes

$$\frac{dn}{dt} - \frac{\rho D^3}{2\pi I}(aV^2+bnDV)+\frac{KV}{2\pi I}=0. \quad . \quad . \quad (4)$$

We now assume that the air-speed follows the sinusoidal law

$$V=V_0(1+\lambda \sin pt),$$

and so, from (4), obtain

$$\frac{dn}{dt}+\alpha(1+\lambda \sin pt)n = \beta(1+\lambda \sin pt)+\gamma(1+\lambda \sin pt)^2, \quad . \quad . \quad . \quad (5)$$

where

$$\left.\begin{aligned} \alpha &= -\frac{\rho b D^4}{2\pi I}V_0, \\ \beta &= -\frac{K}{2\pi I}V_0, \\ \gamma &= \frac{a\rho D^3}{2\pi I}V_0^2. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The solution of (5) is

$$ne^{\alpha\left(t-\frac{\lambda}{p}\cos pt\right)} = \int [(\beta+\gamma)+(\beta+2\gamma)\lambda \sin pt + \gamma\lambda^2 \sin^2 pt]e^{\alpha\left(t-\frac{\lambda}{p}\cos pt\right)}\,dt+C. \quad . \quad . \quad . \quad (7)$$

3. As it stands, this solution can be carried no further, and it is necessary to take into consideration the relative magnitudes of the numerical constants. It seems reasonable to suppose that in many cases of practical importance the period of the pulsations will be faster than one or two per second, and that the amplitude will probably not exceed ten or twenty per cent. Let us assume, therefore, that $\lambda=0{\cdot}1$ and that the period is half a second, i. e., $p=4\pi$. Then $\frac{\lambda}{p}=0{\cdot}008$ sec. approximately, and therefore the maximum value of $\frac{\lambda}{p}\cos pt$ will be 0·008. Hence, after the motion has proceeded for one second the value of $\frac{\lambda}{p}\cos pt$ will be less than 1 per cent. of

$\left(t-\frac{\lambda}{p}\cos pt\right)$, and its relative magnitude will thereafter decrease still further. This suggests that the term $\frac{\lambda}{p}\cos pt$ can be neglected in the index of e in equation (7) in considering the steady state to which the anemometer ultimately settles down.

Admittedly these practical considerations are not sufficient justification for making the assumption that this term can be neglected, and the criticism that has since been advanced against the assumption can only be refuted by the fact that, as is shown below, numerical integration of the complete equation (7) does demonstrate that the term is negligible so far as the average value of n over a complete cycle is concerned. Moreover, the term is negligible not only for the somewhat limited ranges of p and λ mentioned above, but also, as will be shown, for considerably wider ranges which will embrace most practical cases of pulsating flow.

4. Proceeding, then, on the assumption that $\frac{\lambda}{p}\cos pt$ can be ignored, we can integrate (7) to give

$$n=\frac{\beta+\gamma}{\alpha}+\frac{\lambda(\beta+2\gamma)}{\alpha^2+p^2}(\alpha\sin pt-p\cos pt)$$
$$+\frac{\gamma\lambda^2}{2\alpha(\alpha^2+4p^2)}(\alpha^2+4p^2-\alpha^2\cos 2pt-2\alpha p\sin pt)+Ce^{-\alpha t} \quad . \quad . \quad . \quad (8)$$

which shows that the motion consists of a periodic oscillation superimposed on a damped initial disturbance. After a few seconds the damped term becomes very small, and the average value of n over a complete cycle is

$$\bar{n}=\frac{\beta+\gamma}{\alpha}+\frac{\gamma\lambda^2}{2\alpha} \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

By substitution from (6) this becomes

$$n=\frac{K}{\rho bD^4}-\frac{aV_0}{bD}-\frac{a\lambda^2}{2bD}V_0. \quad . \quad . \quad . \quad (10)$$

In steady motion $\lambda=0$, and the residual part of (10) is the equation of the linear portion of the calibration curve of the anemometer. In a pulsating air-stream

λ has a finite value, and the term $-\frac{a\lambda^2}{b\mathrm{D}}\mathrm{V}_0$ represents the amount by which the anemometer over-estimates * the mean air-speed which, in the case considered, is V_0.

5. Measurements of the torque and of the mechanical constants of a particular vane anemometer designed for use at low air-speeds (not exceeding 15 ft./sec.) gave the following values :—

$\mathrm{K}=28\times10^{-7}$ lbs. ft./sec.
$a=0{\cdot}118$.
$b=-0{\cdot}283$.
$\mathrm{D}=0{\cdot}312$ ft.
$\mathrm{I}=192\times10^{-8}$ slugs ft.^{2}

Hence for this anemometer equation (10) becomes

$$n=-0{\cdot}44+1{\cdot}335\,\mathrm{V}_0+1{\cdot}335\frac{\lambda^2}{2}\mathrm{V}_0. \quad . \quad . \quad (11)$$

The experimental calibration curve for this anemometer had the equation

$$n=-0{\cdot}59+1{\cdot}376\,\mathrm{V}_0,$$

which agrees reasonably well with (11) when $\lambda=0$, *i. e.*, in a steady wind. The differences, presumably, arise in the calculation of a, b, and K from the calibration curve.

6. Equation (11) gives the relation between the average air-speed and the average rate of rotation of the vanes in a pulsating air-current. In practice the mean air-speed is derived from the timed rate of rotation by reference to the calibration curve. If in doing this one assumes the calibration curve for a steady condition to be valid also for pulsating flow, a value of the average air-speed V_i will be deduced which exceeds the true time-average of the pulsations. For the case considered the true average is V_0, and it is easy to show from (11) that

$$\mathrm{V}_i=\mathrm{V}_0\left(1+\frac{\lambda^2}{2}\right). \quad . \quad . \quad . \quad . \quad . \quad (12)$$

* The quantity b is negative.

Hence the percentage error on the true average air-speed made by neglecting the over-estimation of the anemometer is

$$E = 100\frac{\lambda^2}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

It is evident from this that a vane anemometer will be seriously in error if the fluctuations of air-speed are at all large.

It will be noted that the solution for the average value of n is independent of the period of the fluctuations. This result, which is no doubt associated with the assumption already discussed, can only be an approximation, for the error must decrease as the period of the pulsations increases, approaching zero when the period is infinite. A rigorous solution would obviously involve both the period of the pulsations and the inertia of the vanes. In the succeeding section the limitations of the approximate solution are discussed numerically.

7. *Numerical Integration of Equation* (7).—In order to compare the approximate solution with the true mean value of n the complete equation (7) was integrated numerically. For this purpose λ was taken as 0·5, *i. e.*, a variation in speed of ± 50 per cent. was assumed, and the period was taken as $\frac{1}{2}$ second. This gave $\frac{\lambda}{p} = 0{\cdot}0398$ sec., so that the neglected term was relatively about five times greater than was assumed in the preceding discussion, and the numerical integration represented a correspondingly more stringent check of the approximate solution.

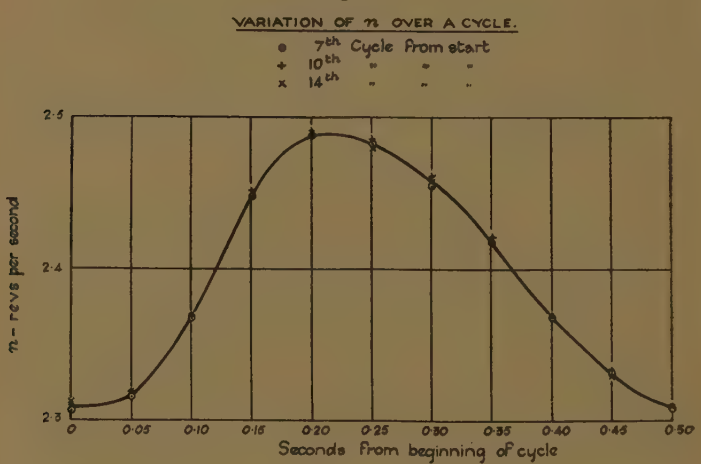
Equation (7), when integrated between the limits 0 and t, becomes

$$ne^{\alpha\left(t-\frac{\lambda}{p}\cos pt\right)} = n_0 e^{-\alpha\frac{\lambda}{p}\cos pt} \int^t [(\beta+\gamma)+(\beta+2\gamma)\lambda \sin pt + \gamma\lambda^2 \sin^2 pt]\, e^{\alpha\left(t-\frac{\lambda}{p}\cos pt\right)}\, dt,$$

where n_0, the value of n when $t=0$, is the value of n corresponding to the steady air-speed V_0.

To simplify the arithmetic V_0 was taken as 1·89 ft./sec., which gave a value of α of 1 sec.$^{-1}$, and the values used for a, b, K and the mechanical constants were those already given for the low-speed anemometer mentioned. The integration was carried out up to $t=7$ seconds, *i. e.*, up to the end of the fourteenth cycle. It was evident, however, that the motion had settled down to a steady oscillation well before this, as will be seen from fig. 1, where the values of n are plotted every tenth of a cycle for the seventh, tenth, and fourteenth cycles.

Fig. 1.



Corresponding values agree within the limits of the numerical accuracy of the integration.

From fig. 1 the mean value of n over a cycle is found to be $\bar{n}=2{\cdot}399$. The approximate solution (equation (11)) gives $\bar{n}=2{\cdot}398$, so that neglect of the term in λ/p is fully justified for values of λ/p up to 0·04 sec.

8. In order to provide some indication of how far the value of λ/p could be increased without serious error, a similar numerical integration was carried out for a sinusoidal variation of V of the same maximum amplitude as before, *i. e.*, $\lambda=0{\cdot}5$, but with the period increased five-fold to $2\frac{1}{2}$ seconds. As has already been

remarked, one would expect the average value of n to be reduced by the longer period. The second graphical integration, which gave a value of $\bar{n}$ of 2·356, confirmed this view, but even for this large increase in the period the value of $\bar{n}$ still differs by less than 2 per cent. from that given by the approximate solution.

Thus, although the approximate solution involves the assumption that the parameter λ/p is small compared with 1 second, actual integration of the complete equation of motion shows that this parameter can be quite large—its value in the second integration was 0·199 sec.—without invalidating the approximate solution to any serious degree. A value of λ/p of 0·2 sec. represents conditions of pulsating flow more extreme than those likely to occur in most practical cases, for although λ may exceed 0·5—the value chosen for the integration—the period will generally be considerably less than $2\frac{1}{2}$ seconds. Hence it appears that we can rely upon the approximate solution in the majority of practical cases in which the velocity variations are sinusoidal or nearly so.

II. Experiments.

9. In order to investigate experimentally the behaviour of the vane anemometer in a current of varying speed an instrument was required as reference that would follow the fluctuations, enable them to be recorded on a time-base, and measure the true mean speed. For this purpose the hot-wire anemometer was the obvious instrument to use, and the writer was fortunate in being able to get the advice of Mr. L. F. G. Simmons, to whom he wishes to record his indebtedness and thanks. Mr. Simmons's experience in the technique of hot-wire measurements was of great value, and it was mainly due to his interest in the problem that a satisfactory method of using the hot wire for these experiments was devised.

10. *Outline of the Method.*—An air-stream of pulsating speed was obtained in a small wind-tunnel by means of a shutter rotating behind a fixed diaphragm and driven from outside the tunnel. Both the diaphragm and the shutter, which consisted of a five-ply disk, had four slits radiating from the centre, so that, four times per revolution of the shutter, four air-passages were alternately blocked and opened together. It was found necessary, however,

in order to get a sufficiently large amplitude of pulsation, for the passages to be closed for about twice as long as they were open. Even so, the amplitude was not as large as had been hoped. The total area of the passages was about 0·14 square foot, and of the unobstructed section of the tunnel 1 square foot.

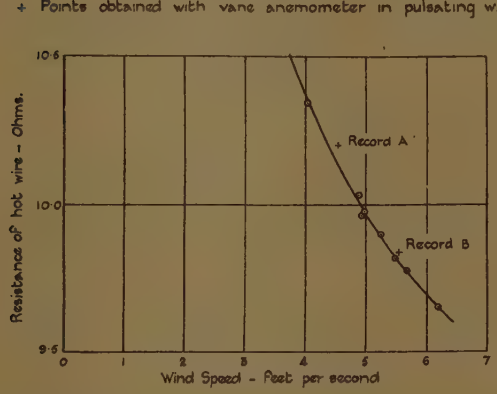
The low-speed vane anemometer already mentioned was supported centrally in the tunnel some distance upstream of the shutter, in a position where, as shown by a few pitot-static traverses, the velocity distribution over an area at least equal to that of the rotor was undisturbed by the shutter and the diaphragm. Three or four inches upstream of the anemometer, on the axis of the tunnel, was the platinum wire, 0·0005 inch diameter and about half an inch long, which was heated from a 100-volt battery through a series resistance of about 800 ohms. A resistance of this high order was chosen so that the current should remain sensibly constant throughout the pulsations, during which the resistance of the wire never varied by as much as half an ohm from its value in a steady wind of the same mean speed. These resistance changes, which were measured by means of a potentiometer in terms of the change of potential across the wire, were used to determine the changes of velocity. For this purpose the wire was connected to an Einthoven galvanometer whose deflexions were recorded photographically. As the full potential drop across the wire would have produced far too large a deflexion of the Einthoven string, the mean potential drop was first balanced out by the aid of the potentiometer, using an ordinary D.C. galvanometer, and only the fluctuations from the mean were recorded by the Einthoven.

The relation between the air-speed and the resistance of the wire was determined by an initial calibration against the vane anemometer in a steady wind, with the shutter full open and not rotating, and it was checked frequently during the investigation, particularly before and after each experiment, in a pulsating wind. Unfortunately the hot-wire calibration changed several times during the series of tests, and it was only possible to obtain two results in which complete confidence could be placed. As will be seen, however, these results agree as closely as could be expected with the theoretical

predictions, and so, in view of the difficulties of the experiment, it was not considered that there was need to spend any more time on the work. It should be mentioned also that altogether nine observations were rejected on account of uncertainty as to the exact hot-wire calibration, but they all provided qualitative confirmation of the theoretical prediction that the anemometer over-estimates the average speed.

11. *Results of Experiments.*—Pl. IX. is a reproduction of one of the Einthoven records. The ordinates are

Fig. 2.



proportional to the current passing through the Einthoven, that is, very nearly proportional to the resistance changes of the hot wire, while the interval between successive peaks is about one-third of a second. A diagram of velocity fluctuations on a time-base would present a similar appearance, although not exactly the same since the relation between the wind-speed and the resistance of the wire was not linear. The peaks of the diagram correspond to the periods of high velocity when the air-passages were open. The truncation of the troughs is due to the fact that when the passages were nominally closed there was considerable leakage between the adjacent faces of the diaphragm and the shutter.

Fig. 2 shows the calibration curve of the hot-wire anemometer in a steady wind, in terms of the resistance of the wire against the air-speed as obtained from the vane anemometer. On the same diagram are shown two points, marked Record A and Record B, obtained in a pulsating wind. The ordinates of these points are the time-averages of the resistance of the wire during the pulsations, as obtained from the electrical observations. They include corrections to allow for the small displacements of the mean height of the Einthoven records from the zero position of the string. These corrections, which were very small, were necessary because the balancing potential (see above) introduced before switching over to the Einthoven did not exactly correspond to the mean resistance of the wire. The abscissæ of the two points in fig 2 were obtained from the observed rate of rotation of the vane anemometer *on the assumption that its calibration was not affected by the pulsations*. The true average air speed corresponding to one of these points is given by the intersection with the hot-wire calibration curve of a horizontal line through the point. It is clear, therefore, that the theoretical prediction that the vane anemometer over-estimates the average speed is confirmed qualitatively by the experiments.

12. *Comparison with Theory.*—For a quantitative comparison between the theory and the experimental results the value of the wind-speed was calculated every tenth of a cycle from the Einthoven diagram of Record A, a mean of corresponding values over four cycles being taken. A smooth curve was drawn through the points so obtained, and the equation of motion was integrated numerically, the velocity V being now written $V = V_0(1+\mu)$ where μ, which replaces $\lambda \sin pt$ of the sinusoidal fluctuation assumed in the theory, was obtained from the smoothed velocity curve. For comparison with this exact result the error on mean velocity was also calculated from the theory as if the velocity variations were sinusoidal with a maximum amplitude equal to the mean of the maximum and minimum observed amplitudes obtained from the Einthoven record. It was found in this way that treatment of the variations as sinusoidal under-estimated the error because of the distortion of the actual wave form. As the velocity diagrams

for Records A and B were very similar in appearance, it was assumed that the ratio of the under-estimation would be the same for Record B as for Record A, and the error calculated for Record B on the assumption of sinusoidal variations was corrected by this ratio in order to avoid repetition of the lengthy graphical integration. The final results of the calculations and the experiments are given in the following table, in which λ_e is the maximum amplitude of the equivalent sinusoidal fluctuation as just defined.

Record.	True mean air-speed (ft./sec.).	Period (secs.).	Equivalent λ (λ_e).	Percentage error of anemometer on mean air-speed.		
				Theoretical.		Observed.
				$\frac{100\lambda_e^2}{2}$.	By actual integration.	
A	4·40	0·353	0·159	1·3	2·3	3·2
B	5·42	0·348	0·186	1·7	3·1*	2·0

13. The results of the last two columns afford the comparison between theory and experiment, and it will be seen that the agreement is satisfactory It is, perhaps, unfortunate that the amplitude of the pulsations was not greater so that the errors to be looked for would have been larger. It is doubtful, however, whether appreciably larger amplitudes could have been obtained by the means adopted, and no other convenient method of producing a pulsating flow suggested itself.

14. *Conclusions.*—The theory and the experiments agree in showing that in a pulsating air-stream a vane anemometer over-estimates the average air-speed. They agree also as regards the magnitude of the error. If the velocity variations can be treated as sinusoidal, of the form $V = V_0(1 + \lambda \sin pt)$, the amount of the error depends on the square of the maximum amplitude. Provided λ/p does not exceed about 0·2 sec., the magnitude of

* Deduced from result for Record A (see above).

the percentage error on mean speed is given with sufficient accuracy by

$$E = \frac{100\,\lambda^2}{2}.$$

The criterion $\lambda/p \not> 0{\cdot}2$ sec. will embrace most practical cases of pulsating flow. In general, the flow will not be sinusoidal, but even then the error will most probably increase roughly as the square of the maximum disturbances from the mean. Hence a vane anemometer should be used with caution to measure air-speed if large pulsations of velocity are suspected.

15. It should be noted that this investigation is restricted to cases of air streams produced by artificial means in which the speed fluctuations will usually be confined mainly to a single frequency associated with the mechanical system used to produce the flow. In natural winds, however, the fluctuations will generally be spread over a wide band of frequencies, and Prof. G. I. Taylor has pointed out that the accuracy of the approximate formula will then depend on the proportion of the total energy of the fluctuations represented by the fluctuations that are faster than those for which one can neglect the term in λ/p. Hence, before the theory could be applied to calculate the behaviour of an anemometer in a natural wind, it would be necessary to analyse typical wind records to find how the energy of the pulsations is distributed over the frequency band.

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XCVI. *Studies in Paramagnetism.*—IV. *On the Ultra-violet Absorption Spectra of Paramagnetic Halides in Solution and the Nature of Chemical Linkage in them.*

By S. Datta *and* Manindra Mohan Deb, *Calcutta University* *.

[Plates X. & XI.]

PREVIOUS investigations by one of us [1] on the magnetic behaviour of different solutions of $CoCl_2$ and $NiCl_2$ in alcohol and water containing excess of HCl at different temperatures have shown that the magnetic properties depend on the nature of the solvent and temperature. The results obtained have been interpreted on the assumption that two different types of aggregates of the paramagnetic ions are formed under different conditions, viz. :—

(i.) Hydrated or alcoholated complexes with six dipole molecules surrounding the paramagnetic ion (at very low temperatures).

(ii.) Undissociated halide molecules held together by covalent bond between paramagnetic atoms and surrounding Cl atoms (at high temperatures).

The paramagnetic susceptibility has always been found to be smaller in the latter type of aggregates as compared to the former, and the calculated magneton number of the ions in the latter class were in better agreement with Bose-Stoner values. Also the Δ-values have been found to increase appreciably in the case of covalent binding [2]. The absorption spectra due to the paramagnetic ions in solutions of $CrCl_3$, $CoCl_2$, and $NiCl_2$ have also been investigated by the present authors [3] under identical conditions in which their magnetic properties were studied.

The paramagnetic ions show strong absorption bands in the visible region, and the positions of those bands have been found to differ considerably in the two types of aggregates mentioned above, the position of the bands being always shifted towards the shorter wave-length side in the case of the first type of aggregates. An explanation of the same has been given, and it has also

* Communicated by Prof. D. M. Bose, M.A., Ph.D.

been shown that the magnetic and optical behaviour of these paramagnetic ions under different conditions can be correlated.

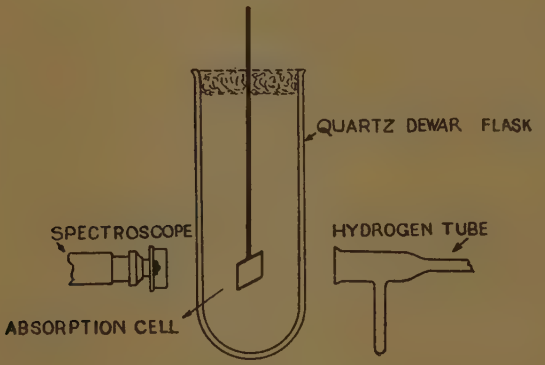
The works of Scheibe and Hantzsch [4] on the absorption spectra of different halogen ions or atoms in halides in the solid or vapour state or in solution have shown that the absorption spectra offer a ready method for studying the nature of the linkage in the halide molecules, and even detecting transition from ionic to covalent type of linkages under varying conditions. With a view to test the different kinds of linkage in the different types of aggregates formed by the paramagnetic ions in different solutions of paramagnetic chlorides under different conditions the present investigations on the absorption spectra in the ultra-violet region due to the halogen ions or atoms in the paramagnetic salts were taken up. The absorption spectra of solutions of $CrCl_3$, $CoCl_2$, and $NiCl_2$ in alcohol, water, and with excess of HCl have been studied at different temperatures (from $+75°$ C. to $-115°$ C.) in the region 4000 Å.U. to 2000 Å.U.

Experimental Investigations.

The source of continuous ultra-violet radiation was a hydrogen discharge-tube 50 cm. long, run at a voltage of 4 kv. and taking a current of 100 ma. Such a tube gives continuous radiation up to about 2000 Å.U., which is the short wave-length limit that can be photographed with a quartz spectrograph. A Fuess quartz spectrograph was used to photograph the spectra in the ultra-violet region. The absorption cell was 2 mm. thick and fitted with windows of plane transparent quartz. For measurements at high or low temperatures the absorption cell was immersed in a specially constructed unsilvered Dewar flask of transparent fused quartz, containing water or ethyl alcohol at different temperatures at which observations are to be made. For observations at higher temperatures the quartz Dewar flask was filled with hot water at the requisite temperature; for low temperature observations the flask was filled with ethyl alcohol, and liquid air was added to the same until the alcohol became a clear viscous glassy liquid, at which stage the temperature of the same was found to remain steady at $-115°$ C. (melting-point of ethyl alcohol).

This was the lowest temperature at which the absorption spectra could be conveniently photographed with the above arrangement. By adjusting the distance of the water- or alcohol-filled Dewar flask from the source and the slit of the spectrograph, a linear image of the source could be focussed on the latter, the flask with its contents serving as a cylindrical lens. The absorption cell, filled with the solution, was immersed in water or alcohol in the Dewar flask, kept in that position for a little time to attain the temperature of the surrounding liquid, and then the absorption spectra were photographed with the quartz Fuess spectrograph. The arrangement is shown

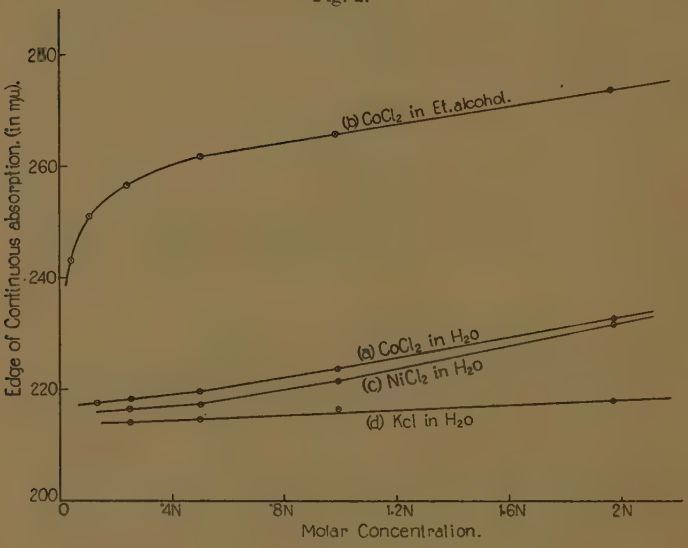
Fig. 1.



in the accompanying diagram (fig. 1). The whole process did not require more than ten minutes, and the temperature of the liquid in the Dewar flask remained constant during this interval. With this arrangement it was possible to photograph the spectra in the ultra-violet region up to the short wave-length limit of the quartz spectrograph at temperatures up to 100° C. when water is used in the quartz Dewar flask. For low temperature observations, however, the Dewar flask being filled with ethyl alcohol, the radiation from the continuous source had to pass through several cm. of the same. At its melting-point a layer of ethyl alcohol of this order of thickness shows continuous absorption after 2380 Å.U., and so it was not possible to investigate the

absorption spectra of the solutions at low temperatures in the still shorter side of that wave-length. Liquid air could not be used as the cooling substance in the Dewar flask in these experiments, as liquid oxygen shows a strong absorption in the ultra-violet region after 2700 Å.U. Any other liquefied gas which would give a temperature below $-100°$ C. and show no absorption in the ultra-violet region is not at present available to us.

Fig. 2.



Results.

In the following tables are given the results obtained with $CoCl_2$, $NiCl_2$, and $CrCl_3$ under different conditions (Tables I.–IV.). For purposes of comparison the ultra-violet absorption spectra of aqueous solution of KCl, which is known to be strongly dissociated, has also been studied, and the results given in Table I. In fig. 2 are shown the edges of continuous absorption at different concentrations in aqueous solutions of $CoCl_2$, $NiCl_2$, and KCl and alcoholic solution of $CoCl_2$. In fig. 3 (*a*) and (*b*) are plotted respectively the edges of continuous absorption in solutions of $CoCl_2$ in water containing varying

TABLE I.

Solution in Water and Ethyl Alcohol. Varying Concentration of Salt at Temperature 30° C.

Salt.	Solvent.	Conc. of salt (molar).	Edge of continuous absorption.	Remarks.
$CoCl_2$.....	H_2O	2 N	2325 Å.U.	See Pl. X. fig. 4 (*a*), and fig. 2 (*a*).
,,	,,	1 N	2240 ,,	
,,	,,	·5 N	2200 ,,	
,,	,,	·25 N	2190 ,,	
,,	,,	·125 N	2180 ,,	
$NiCl_2$.....	,,	2 N	2320 ,,	See Pl. X. fig. 4 (*c*), and fig. 2 (*c*).
,,	,,	1 N	2220 ,,	
,,	,,	·5 N	2180 ,,	
,,	,,	·25 N	2170 ,,	
KCl	,,	2 N	2185 ,,	See fig. 2 (*d*).
,,	,,	1 N	2170 ,,	
,,	,,	·5 N	2155 ,,	
,,	,,	·25 N	2145 ,,	
$CoCl_2$.....	C_2H_5OH	2 N	2745 ,,	See Pl. X. fig. 4 (*b*), and fig. 2 (*b*).
,,	,,	1 N	2665 ,,	
,,	,,	·5 N	2630 ,,	
,,	,,	·25 N	2575 ,,	
,,	,,	·1 N	2520 ,,	
,,	,,	·02 N	2440 ,,	

TABLE II.

Solution in Water containing excess of HCl. Varying Concentration of HCl at Temperature 30° C.

Salt.	Conc. of salt (molar).	Conc. of HCl (molar).	Edge of continuous absorption.	Remarks.
$CoCl_2$.....	·5 N	10·14 N	2665 Å.U.	See Pl. X. fig. 5 (*a*), and fig. 3 (*a*).
,,	,,	8·22 N	2600 ,,	
,,	,,	5·48 N	2460 ,,	
,,	,,	4·11 N	2340 ,,	
,,	,,	2·74 N	2320 ,,	
,,	,,	1·37 N	2300 ,,	
$NiCl_2$.....	,,	8·22 N	2565 ,,	See Pl. X. fig. 5 (*b*).
,,	,,	5·48 N	2305 ,,	
,,	,,	4·11 N	2290 ,,	
,,	,,	2·74 N	2240 ,,	

Table III.

Solutions in Different Solvents at Temperature 30° C.

Salt.	Conc. of salt (molar).	Solvent.	Edge of continuous absorption.	Remarks.
$CoCl_2$.....	1 N	H_2O	2260 Å.U.	See Pl. XI. fig. 6 (*a*).
,,	,,	C_2H_5OH	2666 ,,	
,,	,,	HCl (10·14 N)	2805 ,,	
$NiCl_2$*	·5 N	H_2O	2180 ,,	See Pl. X. fig. 4(*c*), and fig. 5 (*b*).
,,	,,	HCl (8·22 N)	2563 ,,	
$CrCl_3$*	·015 N	H_2O (violet solution)	2170 ,,	See Pl. XI. fig. 6 (*b*).
,,	,,	HCl (8·22 N)	2590 ,,	

* Anhydrous salts could not be dissolved in sufficient quantity in absolutely dry alcohol, and so alcoholic solutions could not be investigated.

Table IV.

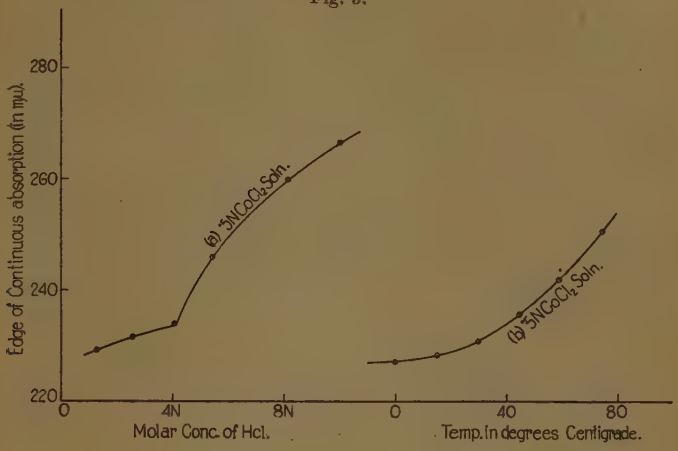
Solutions in Different Solvents at Different Temperatures.

Salt.	Conc. of salt (molar).	Solvent.	Temp.	Edge of continuous absorption.	Remarks.
$CoCl_2$ ..	·5 N	HCl (6·85 N)†	+ 60° C.	2745Å.U.	See Pl. XI. fig. 7(*a*).
,,	,,	,,	−115° C.	2420 ,,	
,,	,,	C_2H_5OH	+ 60° C.	2700 ,,	See Pl. XI. fig. 7(*b*).
,,	,,	,,	−115° C.	2420 ,,	
$NiCl_2$ ..	,,	HCl (6·85 N)†	+ 60° C.	2605 ,,	See Pl. XI. fig. 7(*c*).
,,	,,	,,	−115° C.	2390 ,,	
$CrCl_3$ ..	·03 N	,,	+ 60° C.	2605 ,,	See Pl.XI. fig. 7 (*d*).
,,	,,	,,	−115° C.	2390 ,,	
$CoCl_2$ ..	·5 N	HCl (2·74 N)	0° C.	2275 ,,	See fig. 3 (*b*).
,,	,,	,,	15° C.	2280 ,,	
,,	,,	,,	30° C.	2300 ,,	
,,	,,	,,	45° C.	2350 ,,	
,,	,,	,,	60° C.	2410 ,,	
,,	,,	,,	75° C.	2500 ,,	

† It was not possible to obtain a solid transparent mass at the low temperature for all concentrations of HCl. With this concentration, which was found out by trial, a transparent mass was obtained at the low temperature.

concentration of HCl and in solutions of the same in water containing a fixed concentration of HCl but at varying temperatures.

Fig. 3.



Discussion of Results.

From the results given in Table I. and plotted in fig. 2, it is found that in aqueous solutions of $CoCl_2$ and $NiCl_2$ with increasing concentration the absorption edge gradually shifts towards longer wave-length, (fig. 1 (*a*) and (*c*)), whereas over the same range of concentration the edges of continuous absorption in aqueous solution of KCl, which lie at somewhat shorter wave-length, show very little relative shift (fig. 2 (*d*)). In $CoCl_2$ solution in ethyl alcohol at large dilutions the absorption edge rapidly moves towards the longer wave-length side with increasing concentration up to about ·5 N, after which concentration the shift of the absorption edge towards the longer wave-length side with increasing concentration becomes much less prominent. Also for the same concentration, the absorption edge in the ethyl alcohol solution of $CoCl_2$ is considerably on the longer wave-length side as compared to that in the aqueous solution.

From the results given in Table II., viz., the positions of the absorption-edge for solutions of $CoCl_2$ and $NiCl_2$ in water containing different concentrations of HCl, it is found that with increasing concentration of HCl the

absorption edge shifts at first slowly and after a certain stage, quite rapidly towards the longer wave-length side (fig. 3 (*a*)), the maximum amount of shift when acid concentration is increased from 1 N to 10 N being of the order of several thousand wave numbers.

From Table III., which gives the positions of absorption edge for solutions of $CoCl_2$, $NiCl_2$, and $CrCl_3$ in different solvents, it is found that under identical conditions the positions of the absorption edge in different solvents vary by several thousand wave numbers.

From the positions of the absorption edge for different solutions of $CoCl_2$, $NiCl_2$, and $CrCl_3$ at temperatures $+60°$ C. and $-115°$ C., as given in Table IV., it is found that in all cases at low temperature the absorption edge is shifted farther in the ultra-violet region as compared with the position of the same at higher temperatures. In Table IV. are also given the positions of the absorption edge in a solution of $CoCl_2$ in water containing an excess of HCl at different temperatures from 0° C. to 75° C.; it is found that with increasing temperature the absorption shifts at first slowly and then rapidly towards the longer wave-length side (fig. 3 (*b*)). Also the curve showing the shift of the absorption edge in solutions of $CoCl_2$ with varying concentrations of HCl at constant temperature (fig. 3 (*a*)) is similar to the curve which shows the shift of the absorption edge of $CoCl_2$ dissolved in a given concentration of HCl as function of the temperature (fig. 3 (*b*)). The degree of ionic dissociation in such solutions is a function both of the temperature and of the concentration of Cl^- ions.

We shall now discuss the above results in the light of theoretical interpretations of the absorption spectra of alkali halide molecules in the ultra-violet region, the molecules existing in different states, viz., as vapour and in solution [5]. The result of experimental investigations with NaI vapour and I^- in solution is given in the following table. In column 1 are given the beginning of the continuous absorption, which shows two maxima whose wave-lengths are given in column 2, and the frequency difference between the same in column 3.

The energy corresponding to the wave-length absorbed (column 1) is sufficient to dissociate :

(i.) The NaI molecule into a Na atom and an I atom either in the $^2P_1$ or in the $^2P_2$ state. The heat of disso-

ciation of the NaI molecules have been calculated from the edge of continuous absorption, and it is found to agree well with the value obtained from thermal data.

(ii.) The I^- ion into a free electron and an I atom, either in the $^2P_1$ or in the $^2P_2$ state.

Frank and Scheibe [5] have proposed the following equation to account for the work necessary to separate the attached electron from the I^- ion :

$$h\nu = E + H + P - S - S',$$

where E = electron affinity of an I atom in the state of vapour,
H = work of hydration of an I^- ion,
P = potential energy of six dipole water molecules whose positive ends are all oriented towards the I^- ion,
S = heat of solution of an I atom,
S′ = heat of solution of an electron.

TABLE V.

	I (λ Å.U.).	II (λ Å.U.).		III.	Remarks.
NaI (vapour) ..	3900	3240	2580	8000 cm.$^{-1}$	The calculated value of $\Delta\nu$ ($^2P_2 - ^2P_1$) for I is 7600 cm.$^{-1}$.
I^- (solution) ..	2260	2250	1920	,,	

In case of I^- ions in solution, due to electrostatic forces exerted by the water molecules, an extra amount of work represented by the additional terms on the right side of the above equation is necessary to remove the attached electron from the I^- ion.

The absorption in the ultra-violet region obtained with the paramagnetic chlorides in different solutions is to be ascribed to the Cl ions, which under different conditions form different types of linkage with the paramagnetic ions. Unlike the alkali iodides, in the case of absorption by chloride ions in solution such double absorption maxima have not been observed, due to the fact that $\Delta(P_2 - P_1)$ is only 800 cm.$^{-1}$; the two absorption maxima lie too close together to be resolved.

The results found for the alkali iodides indicate that the absorption edge is shifted considerably to the longer wave-length side in the case of NaI vapour, where I

forms part of an undissociated molecule, as compared to the position of the same for iodide solutions where the I exist as dissociated negative ions ; also the position of the absorption-edge in the latter case depends on several factors represented by the additional terms on the right-hand side of the equation given above. In the case of aqueous solution of KCl, which is a strong electrolyte, for all concentrations investigated (·25 N–2 N) the absorption edges lie on the shorter side of 2200 Å.U. and show very small relative shift. This is to be anticipated, since in KCl solution the Cl^- ions are totally dissociated even at the maximum concentration used in this investigation, so that the terms H, P, S etc. have attained steady values and there is no change with concentration. In the case of the aqueous solutions of the paramagnetic chlorides, however, though at smaller concentrations, the edge of continuous absorption is on the shorter side of 2200 Å.U., the same shifts slightly towards longer wave-length with increasing concentration, indicating thereby that at smaller concentrations these paramagnetic halides are largely dissociated and the continuous absorption is due to the Cl^- ions, and the gradual shift of the absorption edge with concentration being due to the gradually increasing number of undissociated molecules. Increasing concentration of HCl in solutions of $CoCl_2$ and $NiCl_2$ (Table II., fig. 3 (*a*)) shifts the absorption edge considerably towards longer wave-length side, at greater concentrations the edge being shifted through several thousand wave numbers. This must be explained as due to the large increase of undissociated molecules in the latter stages. Considerable work [6] has been done on the absorption spectra of $CoCl_2$ solutions containing excess of HCl, in the visible region, and some of the previous workers have arrived at the conclusion that in such solutions the complex ion $(CoCl_4)^{--}$ is formed. One of us [1, 2] in a previous paper on the magnetic properties of Co^{++} and Ni^{++} halides in solution, has shown that the magnetic behaviour of $CoCl_2$ solutions in excess of HCl can be explained by the formation of undissociated molecules of $CoCl_2$. The same may also explain the absorption spectra in the visible region of $CoCl_2$ solutions containing excess of HCl [3]. In the case of such solutions containing excess of HCl, *i. e.*, containing a large concentration of Cl^- ions, the possibility of formation of complex ions like

$(CoCl_4)^{--}$ always remains; but, according to the present authors even if such complex ions are formed they must be formed by two Cl^- ions covalently linked with the Co^{++} ion, while the other two Cl^- ions are attached to the same by purely electrostatic forces, and, so far as the magnetic properties or the absorption spectra are concerned, it is the covalent linkage between the Co^{++} and the Cl atoms which is of importance and which influences the same. The view is further supported by the fact that, so far as magnetic susceptibility, absorption in the visible and in the ultra-violet region are concerned, there is a remarkable similarity in the behaviour of solutions of $CoCl_2$ dissolved in ethyl alcohol or in an excess of HCl. There is, however, no reason to suppose that such complex ions as $(CoCl_4)^{--}$ are formed in solutions of $CoCl_2$ in ethyl alcohol, as there is no excess of Cl^- ions. The behaviour of $CoCl_2$ solution in alcohol is simply explained by the assumption of formation of undissociated molecules $CoCl_2$.

The influence of solvents on the nature of linkage, as shown by the position of the absorption edge, are seen from the results in Table III. The positions of the absorption edges for solutions in ethyl alcohol and excess of HCl are at considerably longer wave-length side in comparison with corresponding positions in aqueous solutions, as in the former, due to lesser dielectric constant of ethyl alcohol or increasing concentration of Cl^- ions, the paramagnetic chlorides exist predominantly as undissociated molecules held together by covalent linkage.

The influence of temperature on the nature of linkage can be seen from the results in Table IV. It is seen that in a solution of $CoCl_2$ in water with excess of HCl, other things remaining the same, the absorption edge considerably shifts towards longer wave-length merely with increasing temperature, indicating that at higher temperatures undissociated $CoCl_2$ molecules are formed. The nature of the curve showing the relation between the absorption edge and temperature bears striking resemblance to that showing relation between the absorption edge and concentration of HCl in solutions containing varying amounts of HCl. This would lead to the conclusion that the concentration of undissociated $CoCl_2$ molecules in a solution of the same in H_2O containing excess of HCl would depend on two factors, viz., concentration of Cl^-

ions and temperature. Also the results for $CoCl_2$, $CrCl_3$, and $NiCl_2$ solutions in ethyl alcohol or HCl solutions at temperatures $-115°$ C. and $+60$ C. indicate that at higher temperatures the chlorides exist as undissociated molecules, whereas at the low temperatures they largely dissociate into ions. All the above conclusions regarding the nature of linkage in the paramagnetic chlorides arrived at from observations on the ultra violet absorption spectra due to Cl^- ions are fully borne out by previous works on them, carried out by one of us on their magnetic properties and absorption spectra in the visible region, under similar conditions.

Summary.

The ultra-violet absorption spectra due to halogen ions or atoms in paramagnetic halide molecules such as $CrCl_3$, $CoCl_2$, and $NiCl_2$ in different solutions have been investigated with a view to test the different kinds of linkage formed between the paramagnetic ions and the halogen ions under different conditions, viz., at different temperatures and in different solvents.

It has been observed that in aqueous solutions of these salts the edge of continuous absorption is on the shorter side of 220 mμ, and shifts slightly towards longer wave-length side with increasing concentration.

For the same concentration of the salt, and under identical conditions of experiment, when a solvent like ethyl alcohol or HCl solution in water is used the edge of continuous absorption is shifted towards longer wave-length side by several thousand wave numbers.

A study of the absorption spectra of these latter solutions at different temperatures reveal that at very low temperatures the edge of continuous absorption shifts back towards the shorter wave-length side by several thousand wave numbers.

From analogy with the observations of Franck and Scheibe on the ultra-violet absorption spectra of alkali-halide molecules under different conditions, it has been concluded that in the aqueous solutions of paramagnetic chlorides, where the absorption edge is at 220 mμ. the latter must be predominantly dissociated, and the absorption is mainly due to Cl^- ions.

The shift of the absorption edge towards longer wave-length side by thousands of wave numbers in the alcohol

and acid solutions can be explained as due to formation of undissociated halide molecules in such cases.

Similar considerations lead to the conclusion that in ethyl alcohol or HCl solutions, though at higher temperatures, the paramagnetic chlorides exist as undissociated molecules, at very low temperatures they again dissociate into ions, and consequently the absorption edge shifts back towards the shorter wave-length side.

In conclusion, the authors desire to express their thanks to Prof. D. M. Bose for his kind interest and helpful suggestions during the progress of the work.

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XCVII. *On the Integration of the Thermo-Elastic Equations.* *By* J. N. Goodier, *M.A.*, *Ph.D.* *

1. WHEN the temperature $T(x, y, z)$ in an elastic solid is not uniform (or linear in x, y, z), the natural thermal expansion of every volume element is partially restrained by the surrounding material, and a state of stress exists. The difference between the actual strain and the natural expansion (an isotropic strain ϵT†, where ϵ is the coefficient of linear expansion) is related to the stress through Hooke's Law. Using the notation $\widehat{xx}$,

* Communicated by the Author.

† ϵT may be equally regarded as a small change in volume from any cause, provided the elastic constants are not affected.

$\widehat{xy}$ etc. for the stress components, and e_{xx}, e_{xy} etc. for the total strain, we have

$$\widehat{xx}=\frac{E}{1+\sigma}\left[\frac{\sigma}{1-2\sigma}\Delta+e_{xx}-\frac{1+\sigma}{1-2\sigma}\epsilon T\right], \quad \widehat{xy}=\mu e_{xy} \text{ etc.,} \quad (1)$$

where $\Delta=e_{xx}+e_{yy}+e_{zz}$, E is Young's modulus, σ Poisson's ratio, and μ the shear modulus. E, σ, and μ are regarded as independent of the temperature.

The equations of equilibrium, with (1), lead to the three differential equations satisfied by the components of the displacement, u, v, w:

$$\frac{\partial\Delta}{\partial(x, y, z)}+(1-2\sigma)\nabla^2(u, v, w)=2(1+\sigma)\epsilon\frac{\partial T}{\partial(x, y, z)}, \quad (2)$$

∇^2 being the Laplacian operator.

Equations (1) and (2) are the fundamental equations of the theory. The boundary condition is most naturally taken as freedom from traction.

Besides involving the limitations of the ordinary theory of elasticity, (1) and (2) neglect thermodynamic effects, which are very small, and treat the elastic constants and the coefficient of expansion as independent of the temperature. The last feature, although the most objectionable, is not in fact a very serious departure from physical reality. Thermal stress is of the order $E\epsilon T$ (the magnitude of the longitudinal stress required to suppress the expansion of a bar for a temperature rise T), where T in general is the maximum temperature difference existing in the body. In steel a temperature difference of only 150° C. may give rise to a stress of about 15 tons per square inch. which would exceed the elastic limit of mild steel. For a similar temperature range, beginning at ordinary temperatures, the elastic constants vary only by some 5 per cent. The coefficient of expansion may vary by about 15 per cent., but this does not affect the accuracy of the theory because the coefficient appears only in the product ϵT, and this product may be carried through the equations as a single quantity without altering the results. We may treat ϵ as a constant if it is understood that T is not necessarily the actual temperature, but a modified temperature such that ϵT has its true value at every point.

The introduction of welding and the higher temperatures, pressures, and precision of modern engineering lend

practical interest to a wide range of problems, but the theory has not been carried very far. The most important existing solutions, for the cylinder and the sphere with temperatures in each case a function of the radius only, were first derived by J. M. C. Duhamel [1] in 1838, and have since found application to the problems of engine cylinders and boiler tubes.

In the present paper a method is developed which, besides leading to certain general theorems, reduces the problem of a given solid with a given temperature distribution to a problem of given surface tractions in the ordinary theory of elasticity. It will be apparent that this can be done in more than one way, but the procedure adopted enables us to take advantage of the theory of potential and the theory of heat conduction.

2. Let

$$u, v, w = \frac{\partial \phi}{\partial x}, \quad \frac{\partial \phi}{\partial y}, \quad \frac{\partial \phi}{\partial z}.$$

Then (2) are satisfied if

$$\nabla^2 \phi = m\mathrm{T}, \qquad \ldots\ldots \quad (3)$$

where m stands for $\epsilon(1+\sigma)/(1-\sigma)$.

Evidently any function ϕ which satisfies (3) provides a particular integral of (2). The state of stress and strain it represents will require for its maintenance not only the given distribution of temperature, but also certain boundary tractions, which can be calculated from it by means of (1) and the conditions of equilibrium at the boundary. The remainder of the problem consists in the determination of a complementary stress distribution due to equal and opposite boundary tractions. This, however, is merely a problem of given boundary tractions in the ordinary theory of elasticity, the fact that the body is heated being immaterial so long as the elastic constants are unaffected.

The function ϕ will be called the (thermo-elastic) displacement potential.

Duhamel's formulæ for the cylinder and sphere follow at once from (3) on specializing to the radial coordinate and integrating. It is only necessary to add, as the complementary solution making the boundaries free, the well-known Lamé solutions for boundary pressures.

Arbitrary Temperature Distribution.

3. The equation (3) is of the same form as Poisson's equation, and a particular integral is given by the Newtonian potential of a distribution of matter of density $-m\mathrm{T}/4\pi$. Then

$$\phi = -\frac{m}{4\pi}\iiint \mathrm{T}(\xi, \eta, \zeta)\,\frac{1}{r'}\,d\xi\,d\eta\,d\zeta, \quad . \quad . \quad . \quad (4)$$

where $r' = [(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2]^{\frac{1}{2}}$,

and the integration is throughout the solid. When we are concerned with local inequalities of temperature, this displacement potential will give displacements which vanish at infinity like r^{-2} and stresses which vanish like r^{-3}. It represents, then, the complete solution for the infinite solid.

4. *Discontinuous Temperature Distribution.*—The general equations (2) imply the the existence of the first derivatives of the temperature. The validity of any solution at a surface S of temperature discontinuity therefore requires examination. We consider first the displacement potential of the infinite solid (4).

Let the subscript i refer to the limit of a function when the surface is approached from one side, the interior, and the subscript e to the limit when S is approached from the other side, the exterior. The temperature discontinuity may then be written $\mathrm{T}_i - \mathrm{T}_e$, a function of position on S.

At any point P on S take right-handed rectangular Cartesian coordinates α, β, γ—α, β lying in the tangent plane and γ along the outward normal.

The conditions which must be satisfied if (4) is to be valid in spite of the temperature discontinuity are that on S the displacement must be continuous and the surface traction must be continuous. These are equivalent to: (i.) the first derivatives of ϕ must be continuous, and (ii.) the stress components of the types $\widehat{\gamma\gamma}$, $\widehat{\alpha\gamma}$, $\widehat{\beta\gamma}$ must be continuous.

Now ϕ, as given by (4), is the potential of distribution $-m\mathrm{T}/4\pi$. It is shown in works on the theory of potential that a distribution ρ which is bounded and integrable has a potential such that

(A) its first derivatives are continuous everywhere, including surfaces where ρ is discontinuous;

(B) its second derivatives are continuous *except* at surfaces where ρ is discontinuous. With coordinates α, β, γ defined as above

$$\left(\frac{\partial^2\phi}{\partial\gamma^2}\right)_i - \left(\frac{\partial^2\phi}{\partial\gamma^2}\right)_e = -4\pi(\rho_i - \rho_e),$$

and all the second derivatives, other than $\partial^2\phi/\partial\gamma^2$, are continuous.

By (A) the displacement derived from (4) will be continuous.

In order to examine the stress components we express them in terms of ϕ. We have as in (1)

$$\widehat{\alpha\alpha} = 2\mu\left[\frac{\sigma}{1-2\sigma}\Delta + e_{\alpha\alpha} - \frac{1+\sigma}{1-2\sigma}\epsilon\mathrm{T}\right], \quad \widehat{\alpha\beta} = \mu e_{\alpha\beta},$$

with similar formulæ for the other components. Now

$$e_{\alpha\alpha} = \frac{\partial u_\alpha}{\partial\alpha} = \frac{\partial^2\phi}{\partial\alpha^2}, \quad e_{\alpha\beta} = \frac{\partial u_\alpha}{\partial\beta} + \frac{\partial u_\beta}{\partial\alpha} = 2\frac{\partial^2\phi}{\partial\alpha\partial\beta},$$

and

$$\Delta = \nabla^2\phi = m\mathrm{T} \quad \text{by (3)}.$$

Then

$$\widehat{\alpha\alpha} = 2\mu\left(\frac{\partial^2\phi}{\partial^2\alpha} - m\mathrm{T}\right), \quad \widehat{\alpha\beta} = 2\mu\frac{\partial^2\phi}{\partial\alpha\,\partial\beta},$$

so that

$$\widehat{\gamma\gamma}_i - \widehat{\gamma\gamma}_e = 2\mu\left[\left(\frac{\partial^2\phi}{\partial\gamma^2}\right) - \left(\frac{\partial^2\phi}{\partial\gamma^2}\right)_e - m(\mathrm{T}_i - \mathrm{T}_e)\right].$$

By the theorem (B) above

$$\left(\frac{\partial^2\phi}{\partial\gamma^2}\right)_i - \left(\frac{\partial^2\phi}{\partial\gamma^2}\right)_e = m(\mathrm{T}_i - \mathrm{T}_e),$$

and consequently $\widehat{\gamma\gamma}_i - \widehat{\gamma\gamma}_e$ vanishes; that is, $\widehat{\gamma\gamma}$ is continuous on S. So also are all the shearing components of the stress. *The solution* (4) *is therefore valid when* T *has surface discontinuities.*

5. The stress components of the types $\widehat{\alpha\alpha}$, $\widehat{\beta\beta}$ on the other hand need not be, and in fact are not, continuous. We have

$$\widehat{\alpha\alpha}_i - \widehat{\alpha\alpha}_e = 2\mu\left[\left(\frac{\partial^2\phi}{\partial\alpha^2}\right)_i - \left(\frac{\partial^2\phi}{\partial\alpha^2}\right)_e - m(\mathrm{T}_i - \mathrm{T}_e)\right]$$
$$= -2\mu m(\mathrm{T}_i - \mathrm{T}_e),$$

and similarly $\widehat{\beta\beta}_i - \widehat{\beta\beta}_e = -2\mu m\ (T_i - T_e)$. It follows that any component of direct stress, acting on an element of area normal to, and at, the surface of discontinuity is discontinuous by an amount $-E\epsilon(T_i - T_e)/(1-\sigma)$.

6. All these conclusions remain true if the solid has finite boundaries. The boundary tractions corresponding to (4) are to be cancelled by superposing an isothermal stress distribution having equal and opposite boundary values. But any distribution of boundary tractions, whether continuous or not, produces a stress which is continuous everywhere in the interior. The discontinuity of stress due to temperature discontinuity remains unaffected, although the absolute values of the stress on either side of S will in general be modified.

It is not necessary that S should lie entirely within the boundary. It may cut in it one or more curves. When it does, the complementary surface tractions are discontinuous at these curves. When it does not, they are continuous all over the boundary.

7. In a shrink fit the two parts fit exactly when one is at a (uniform) temperature T_e, the other at T_i, and the composite body is then free from stress. If the two parts adhere at the common surface, so that no slip can occur during the equalization of temperature, the state of stress is finally the same as if the body were of one piece, but the inner part heated to a temperature $T_e - T_i$. The stress discontinuity at the common surface (S) is therefore given by

$$\widehat{\alpha\alpha}_i - \widehat{\alpha\alpha}_e = \widehat{\beta\beta}_i - \widehat{\beta\beta}_e = -\frac{E\epsilon}{1-\sigma}(T_e - T_i),$$

whatever the shape of S.

8. Thermal stress is essentially due to non-uniformity, not of temperature alone, but of the product ϵT. If a composite solid is formed of two parts of the same elastic properties, but different coefficients of expansion, ϵ_i and ϵ_e, at temperature zero, and the temperature is then changed to T, there will be a stress discontinuity

$$\widehat{\alpha\alpha}_i - \widehat{\alpha\alpha}_e = \widehat{\beta\beta}_i - \widehat{\beta\beta}_e = -\frac{ET}{1-\sigma}(\epsilon_i - \epsilon_e)$$

at the common surface. In general the stress throughout is the same as it would be in a solid with a coefficient of

expansion ϵ_0, uniform throughout, heated to a non-uniform temperature (ϵ/ϵ_0)T, ϵ being the actual coefficient regarded as a function of position.

9. It is now possible to define the nucleus of thermo-elastic strain. The formula (4) and the results of Art. 4 show that if an element of volume $d\tau$ in the infinite solid is at a temperature T, the remainder being at temperature zero, the displacement is the gradient of

$$-\frac{m\mathrm{T}}{4\pi}d\tau.\frac{1}{r'}.$$

This is simply the singularity known in the ordinary theory of elasticity as the "centre of dilatation," and $m\mathrm{T}d\tau/4\pi$ may be called its strength. Evidently the effect of non-uniform heating is the same as that of a distribution of centres of dilatation of this strength in an unheated body.

By means of this result we may write down a complete solution of the thermo-elastic problem of a bounded solid, if the formulæ for the centre of dilatation within the boundary are known. For these formulæ will consist of a centre of dilatation in the infinite solid, together with functions representing the effects of removing the tractions on the given boundary, and, therefore, regular within the boundary. We have only to assign a strength which will make the strength of the singular part $m\mathrm{T}d\tau/4\pi$ and integrate.

There is no difficulty in applying this method to the circular cylinder (as a two-dimensional problem) and the sphere, but the forms we should obtain have already been found by C. W. Borchardt [2] by methods peculiar to these shapes. In general, however, the method of singularities is not so appropriate as the method of series when numerical results are sought. The latter method is applied below to the problem of the circular cylinder.

Displacement Potentials consistent with the Law of Heat Conduction.

10. Excepting the arbitrary initial distribution, the temperatures in elastic solids are naturally governed by the law of heat conduction. In the steady state T is a harmonic function, if there is no internal generation of heat. Then, by (3), ϕ is a bi-harmonic function, and

its expression will depend on the shape of the boundary. When, however, the temperature is a function of time *, as well as position, the integration of (3) can be very simply effected in general terms.

The equation of heat conduction, including internal generation of heat at a rate q per unit volume, is

$$\frac{\partial \mathrm{T}}{\partial t} = \kappa \nabla^2 \mathrm{T} + \frac{q}{h\rho}, \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where t is the time, κ the thermal diffusivity (or the conductivity K divided by $h\rho$), h the specific heat, and ρ the density.

Differentiating (3) with respect to t, and eliminating $\partial \mathrm{T}/\partial t$ by means of (5),

$$\nabla^2 \frac{\partial \phi}{\partial t} = m\left(\kappa \nabla^2 \mathrm{T} + \frac{q}{h\rho}\right).$$

The complete solution of this is

$$\frac{\partial \phi}{\partial t} = m(\kappa \mathrm{T} + \mathrm{Q}) + \chi(x, y, z),$$

where Q is any solution of $\nabla^2 \mathrm{Q} = q/h\rho$, in general a function of x, y, z, and t, and $\chi(x, y, z)$ is harmonic in x, y, z, and independent of t.

Integrating with respect to t,

$$\phi = m \int_{t_1}^{t} (\kappa \mathrm{T} + \mathrm{Q})\, dt + t\chi(x, y, z) + \psi(x, y, z), \quad . \quad (6)$$

where t_1 is an arbitrary initial instant and ψ is, so far, an arbitrary function of integration. This result has been obtained, however, after differentiation of (3). On substitution it is found to satisfy (3) itself provided that ψ is any solution of $\nabla^2 \psi = m\mathrm{T}_1$, where T_1 is the temperature distribution at the instant t_1.

The significance of the term $t\chi(x, y, z)$ is illustrated by considering a system which tends to a steady state, T, Q, and q, becoming independent of t after an indefinitely long time. Then ϕ must also be independent of further lapse of time. This is secured by making

$$\chi(x, y, z) = -m \lim_{t \to \infty} (\kappa \mathrm{T} + \mathrm{Q}) = -m(\kappa \mathrm{T}_\infty + \mathrm{Q}_\infty).$$

* Dynamical effects of changing temperatures are neglected, the propagation of temperature being normally extremely slow in comparison with the propagation of stress.

It is then also convenient to take t_1 as infinite, so that ψ will be a solution of $\nabla^2\psi = m\mathrm{T}_\infty$.

11. A simpler form of displacement potential is obtained when the solid is cooling, or being heated, to a uniform temperature, with no internal generation of heat. In either process let the final temperature be taken as zero. Then it is sufficient to take $\chi = \psi = 0$, and

$$\phi = -m\kappa \int_t^\infty \mathrm{T}\, dt. \quad . \quad . \quad . \quad . \quad . \quad (7)$$

In the application of the method of series to the solution of the conduction equation

$$\frac{\partial \mathrm{T}}{\partial t} = \kappa \nabla^2 \mathrm{T} \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

the temperature is taken as a series of the form

$$\sum_s e^{-\kappa s^2 t} f_s(x, y, z),$$

the functions f_s being then solutions of the set of equations $(\nabla^2 + s^2) f_s = 0$.

When the process is one of cooling or heating to a steady state, the numbers s^2 are real and positive. Each term of the temperature series will then fulfil the conditions attached to (7), and, therefore, there is a displacement potential

$$\phi = \Sigma \phi_s = -\sum_s \frac{m}{s^2} e^{-\kappa s^2 t} f_s(x, y, z). \quad . \quad . \quad . \quad (9)$$

The appropriate forms for cylindrical, spherical, and other boundaries, corresponding to very general thermal conditions, can be written down at once from the well-known series for the temperatures [3].

The complementary stress (cancelling the surface stress of (9)) for cylinders with temperatures independent of the axial coordinate, is easily found from the general forms of the Airy stress-function in polar or other curvilinear coordinates. The circular case is treated below. If the temperature is symmetrical about the axis, but varies in the axial direction, appropriate forms are also known. In the case of the sphere, Kelvin's solution of the problem of given surface tractions is available as soon as the surface stress of the displacement potential has been expressed in terms of surface harmonics.

12. In questions of fluctuating temperatures, or propagation of temperature waves, we encounter the typical form

$$T_\omega = P_\omega(x, y, z) \cos \omega t + Q_\omega(x, y, z) \sin \omega t, \quad . \quad (10)$$

which may be written as the real part of $(P_\omega + iQ_\omega)e^{-i\omega t}$. The conduction equation (8) then requires that

$$\nabla^2(P_\omega + iQ_\omega) = -\frac{i\omega}{\kappa}(P_\omega + iQ_\omega). \quad . \quad . \quad (11)$$

The corresponding displacement potential may be found from (6), but is more conveniently derived *a priori*. It will have the same frequency of fluctuation as T_ω, and may therefore be taken as

$$\phi_\omega = F_\omega(x, y, z) \cos \omega t + G_\omega(x, y, z) \sin \omega t,$$

or as the real part of $(F_\omega + iG_\omega)e^{-i\omega t}$. Equation (3) requires that

$$\nabla^2(F_\omega + iG_\omega) = m(P_\omega + iQ_\omega).$$

But this is satisfied, on account of (11), if

$$F_\omega + iG_\omega = \frac{im\kappa}{\omega}(P_\omega + iQ_\omega).$$

The displacement potential corresponding to (10) will then be

$$\phi_\omega = \frac{m\kappa}{\omega}[P_\omega(x, y, z) \sin \omega t - Q_\omega(x, y, z) \cos \omega t],$$

and can thus be written down at once from the temperature.

Two-Dimensional Systems.

13. *Plane Strain.*—In the ordinary isothermal theory of elasticity, plane strain is defined by the conditions that $e_{zz} = 0$, and all displacements and stresses are independent of the axial coordinate z.

The maintenance of such a state requires that the applied tractions, on the cylindrical boundaries (with generators parallel to the z-axis) should be independent of z, and it requires in each case a certain distribution of normal traction on the plane ends $z =$ constant, whether these be at infinity or not.

The conditions required for plane thermal strain are simply that the temperature should be independent of

the axial coordinate z, and that certain end tractions should act, as required by the temperature distribution.

The end tractions can be resolved into a uniform part, and a non-uniform part constituting a resultant couple. The removal of the resultant couple will produce flexure, and the strain will no longer be plane. It is therefore convenient, in problems concerning long cylinders and temperatures independent of the axial coordinates, to suppose that the end tractions exist, so that the problem can be simplified to one of plane thermal strain. The effects of removing the end tractions can be considered afterwards.

The relevant equations are exactly as in the general case, but all terms involving z, with the exception of the stress-component $\widehat{zz}$, drop out.

The solution (4) still holds, but the integral can be replaced by the equivalent logarithmic integral

$$\phi = \frac{m}{2\pi}\iint \mathrm{T}(\xi, \eta) \log r' \, d\xi \, d\eta,$$

where $r' = [(x-\xi)^2 + (y-\eta)^2]^{\frac{1}{2}}$, and the integration now extends over the ξ–η-plane. This differs from (4) by an infinite constant.

The arguments of Arts. 4–8 apply without change.

All the remaining general conclusions hold, under the condition that all functions are functions of x and y only.

The plane-strain problem of greatest practical interest is probably that of the circular cylinder. The case of temperature a function of radius only is easily dealt with by means of Duhamel's formula.

The results we have obtained lead to series formulæ for the general case of plane strain, when the temperature is not symmetrical about the axis.

The typical term in the temperature series is (3)

$$\mathrm{T}_{sn} = e^{-\kappa s^2 t} \mathrm{J}_n(sr) e^{in\theta},$$

where the real or imaginary part may be taken to obtain $\cos n\theta$ or $\sin n\theta$. Corresponding to this we shall have as a possible displacement potential $\phi_{sn} = -\frac{m}{s^2}\mathrm{T}_{sn}$, by (9). The associated stress components are given by

$$\widehat{rr} = 2\mu\left(\frac{\partial^2\phi}{\partial r^2} - m\mathrm{T}\right), \quad \widehat{r\theta} = 2\mu\,\frac{\partial}{\partial r}\left(\frac{1}{r}\,\frac{\partial\phi}{\partial\theta}\right)$$

$$\widehat{\theta\theta}=2\mu\Big(\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}-m\mathrm{T}\Big),\quad \widehat{zz}=-2\mu m\mathrm{T},$$

whence

$$\left.\begin{aligned}\widehat{rr}_{sn}&=-2\mu m e^{-\kappa s^2 t}(sr)^{-2}[n(n-1)\mathrm{J}_n(sr)+sr\mathrm{J}_{n+1}(sr)]e^{in\theta},\\ \widehat{r\theta}_{sn}&=-2\mu m e^{-\kappa s^2 t}(sr)^{-2}[(n-1)\mathrm{J}_n(sr)-sr\mathrm{J}_{n+1}(sr)]ie^{in\theta},\end{aligned}\right\}\quad . \quad . \quad . \quad (12)$$

and $\widehat{\theta\theta}_{sn}=-2\mu m e^{-\kappa s^2 t}(sr)^{-1}[sr\mathrm{J}_n(sr)-\mathrm{J}_{n+1}(sr)]e^{in\theta}.$

We have now to find the complementary stress distribution cancelling the stress (12) on the boundary. For this we may introduce the Airy stress function given by a series of terms such as

$$\chi_{sn}=(a_{sn}r^n+b_{sn}r^{n+2})e^{in\theta},$$

the real or imaginary part pertaining to the corresponding part of the temperature.

The stress associated with χ_{sn} is given by well-known formulæ. We have

$$\left.\begin{aligned}\widehat{rr}'_{sn}&=\frac{1}{r}\frac{\partial\chi_{sn}}{\partial r}+\frac{1}{r^2}\frac{\partial^2\chi_{sn}}{\partial\theta^2}\\ &=-[n(n-1)a_{sn}r^{n-2}+(n+1)(n-2)b_{sn}r^n]e^{in\theta},\\ \widehat{r\theta}'_{sn}&=-\frac{\partial}{\partial r}\Big(\frac{1}{r}\frac{\partial\chi_{sn}}{\partial\theta}\Big)\\ &=-[(n-1)a_{sn}r^{n-2}+(n+1)b_{sn}r^n]ine^{in\theta},\end{aligned}\right\}\quad . \quad (13)$$

$$\widehat{\theta\theta}'_{sn}=\frac{\partial^2\chi_{sn}}{\partial r^2}=[n(n-1)a_{sn}r^{n-2}+(n+2)(n+1)b_{sn}r^n]e^{in\theta}.$$

The condition determining a_{sn} and b_{sn} is that (12) and (13) should be equal and opposite on the boundary $r=\lambda$. The solution of the two simple equations so obtained is

$$a_{sn}=-\frac{\mu m}{s^2\lambda^n}[2\mathrm{J}_n(s\lambda)+s\lambda\mathrm{J}_{n+1}(s\lambda)]e^{-\kappa s^2 t},$$

$$b_{sn}=\frac{\mu m}{s\lambda^{n+1}}\mathrm{J}_{n+1}(s\lambda)e^{-\kappa s^2 t}.$$

The actual stress components due to T_{sn} are now completely determined as the sum of those derived from ϕ_{sn} and χ_{sn}. The longitudinal tension $\widehat{zz}_{sn}$ consists of the part $-2\mu m\mathrm{T}_{sn}$ from the ϕ_{sn} system and a part, $\sigma(\widehat{rr}'_{sn}+\widehat{\theta\theta}'_{sn})$, from the χ_{sn} system. The effects of removing the corresponding end actions may be found, except for the terminal

irregularity, by the well-known formulæ for the stretching and bending of rods.

14. *Plane Stress.*—In the isothermal theory plane stress is defined, for a thin plate of uniform thickness with faces parallel to the $x-y$-plane, by the conditions $\widehat{zz}=\widehat{xz}=\widehat{yz}=0$ throughout *. Equations are derived for the values of the displacements u, v and the stresses $\widehat{xx}$, $\widehat{xy}$, $\widehat{yy}$.

When the stress is due to heating, the same assumptions can be adopted, T being a function of x and y only.

The stress-strain relations take the form

$$\widehat{xx}=\frac{\mathrm{E}}{1-\sigma^2}\left[\frac{\partial u}{\partial x}+\sigma\frac{\partial v}{\partial y}-(1+\sigma)\epsilon\mathrm{T}\right],$$

$$\widehat{yy}=\frac{\mathrm{E}}{1-\sigma^2}\left[\frac{\partial v}{\partial y}+\sigma\frac{\partial u}{\partial x}-(1+\sigma)\epsilon\mathrm{T}\right],$$

$$\widehat{xy}=\frac{\mathrm{E}}{2(1+\sigma)}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right).$$

The two outstanding equations of equilibrium then lead to

$$\frac{\partial \Delta_1}{\partial(x,y)}+\frac{1-\sigma}{1+\sigma}\nabla_1^2(u,v)=2\epsilon\frac{\partial \mathrm{T}}{\partial(x,y)},$$

where

$$\Delta_1\equiv\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y},\quad \nabla_1^2\equiv\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}.$$

A particular integral is given by $u,\ v=\dfrac{\partial\phi}{dx},\ \dfrac{\partial\phi}{\partial y}$ if

$$\nabla_1^2\phi=(1+\sigma)\epsilon\mathrm{T}. \quad . \quad . \quad . \quad . \quad . \quad (14)$$

The constant $(1+\sigma)\epsilon$ will be abbreviated to m_1.

The solution corresponding to (4) is now the potential of a distribution of matter of space density $-m_1\mathrm{T}/4\pi$, a function of x and y only, and this is given by the logarithmic potential

$$\frac{m_1}{2\pi}\iint\mathrm{T}(\xi,\eta)\log r'\,d\xi\,d\eta. \quad . \quad . \quad . \quad (15)$$

The theory corresponds to that of plane strain, the constant m_1 replacing m. At a temperature discontinuity, along a curve to which β is the normal, α the tangent, the

* The small z-variations implied by these conditions will be ignored.

stress components $\widehat{\beta\beta}$, $\widehat{\alpha\beta}$ are continuous, but $\widehat{\alpha\alpha}$ is discontinuous, being given by $2\mu\left(\frac{\partial^2\phi}{\partial\alpha^2} - m_1 T\right)$, so that $\widehat{\alpha\alpha}^i - \widehat{\alpha\alpha}_e = -\mathrm{E}\epsilon(\mathrm{T}_i - \mathrm{T}_e)$ in virtue of the continuity of $\partial^2\phi/\partial\alpha^2$.

15. There are two problems of the heated plate which are mathematically simple, and are interesting in connexion with the influence of the shape of the hot region on the maximum stress. In the first an elliptical area is at a uniform temperature T, and the rest of the (infinite) plate is at zero temperature. In the second the hot area is rectangular. The former may be expected to correspond to regular elongated hot areas, and the latter will show the effects of sharp corners.

The Elliptical Area.—Corresponding to (15) we have the displacement potential, in the elliptic coordinates defined by $x+iy=c\cosh(\xi+i\eta)$,

$$\phi = \tfrac{1}{4}(1+\sigma)\epsilon \mathrm{T}\, ab\,(e^{-2\xi}\cos 2\eta + 2\xi),$$

where a, b are the semi-axes of the ellipse $\xi=\xi_0$ bounding the hot area. This holds for the cold exterior, $\xi>\xi_0$, where $\widehat{xx}=2\mu\partial^2\phi/\partial x^2$ etc. It follows that the stress component of the type $\widehat{\eta\eta}$ (the "hoop stress" round the periphery of an ellipse) is given by

$$2\mu h^2\left[\frac{\partial^2\phi}{\partial\eta^2} - \frac{1}{2h^2}\frac{\partial h^2}{\partial\xi}\frac{\partial\phi}{\partial\xi} + \frac{1}{2}\frac{1}{h^2}\frac{\partial h^2}{\partial\eta}\frac{\partial\phi}{\partial\eta}\right],$$

where $h^2=2/c^2(\cosh 2\xi - \cos 2\eta)$. Then

$$\widehat{\eta\eta} = \mathrm{E}\epsilon\mathrm{T}.h^2 ab\left\{-e^{-2\xi}\cos 2\eta + \frac{c^2h^2}{8}\left[e^{2\xi} - (1-e^{-1\xi})\cos 2\eta - e^{-2\xi}\cos 4\eta\right]\right\}.$$

Just beyond the hot area, on the major axis ($\xi=\xi_0$, $\eta=0$) this is a tensile stress $\mathrm{E}\epsilon\mathrm{T}/(1+b/a)$. If the hot area is very elongated this will be nearly $\mathrm{E}\epsilon\mathrm{T}$, whereas if the area is circular it is $\frac{1}{2}\mathrm{E}\epsilon\mathrm{T}$. Just beyond the end of the minor axis ($\xi=\xi_0$, $\eta=\pi/2$) $\widehat{\eta\eta}$ is also tensile, being $\mathrm{E}\epsilon\mathrm{T}/(1+a/b)$, and becomes small if the ellipse is elongated.

The Rectangular Area.—The second derivatives of the logarithmic potential of a rectangle are easily calculated directly. For the cold region we find

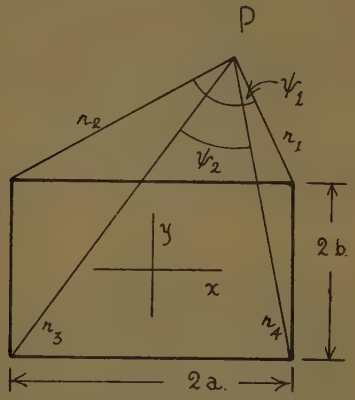
$$\widehat{xx} = \mathrm{E}\epsilon\mathrm{T}\frac{\psi_1 - \psi_2}{2\pi}$$

at any point P, ψ_1 and ψ_2 being the angles indicated in the figure. This is a tensile stress approaching

$$\frac{E\epsilon T}{2\pi}\left(\frac{\pi}{2}+\tan^{-1}\frac{b}{a}\right)$$

in the neighbourhood of a corner. The greatest value, $\frac{1}{2}E\epsilon T$, is attained when the rectangle is elongated.

The normal components of stress are evidently discontinuous at the corner, not only as the boundary is crossed, but also as the corner is turned. It is to be expected, therefore, that there will be sudden variations, even if the corner is slightly rounded, and this would imply high shear stress.



The value of the shear stress for the perfect rectangle is

$$\widehat{xy}=\frac{E\epsilon T}{4\pi}\log\frac{r_1 r_3}{r_2 r_4},$$

and is infinite at the corners.

16. If a thin plate is losing heat by emission from the surface at the rate νT per unit area, as by the Newtonian law of cooling, the equation of heat conduction is

$$K\nabla_1^2 T=\nu T, \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

the temperature of the atmosphere being taken as zero.

Comparing (16) with (14), it is clear that the displacement potential may be taken simply as

$$\phi=K\frac{m_1}{\nu}T=K(1+\sigma)\frac{\epsilon}{\nu}T. \quad . \quad . \quad . \quad . \quad (17)$$

Consider, for instance, a large plate in which a source of heat at the origin supplies heat H per unit time. The appropriate solution of (16) is then

$$T=\frac{H}{2\pi dK}\cdot K_0(lr),$$

where $l=(\nu/K)^{\frac{1}{2}}$, d is the plate thickness and K_0 is the Bessel function of imaginary argument, vanishing at infinity [4]. The displacement potential follows from (17). The complete radial displacement is

$$u=\frac{\epsilon(1+\sigma)H}{2\pi d\nu}\left[lK_1(lr)+\frac{1}{r}\right].$$

The complementary displacement proportional to r^{-1} has been introduced to cancel the singularity of $lK_1(lr)$ at the origin, which, being of order higher than that of the temperature itself, is physically inadmissible.

Summary.

Solutions are developed of the equations for stress and strain, in an elastic solid, due to uneven heating, the temperatures being either arbitrary or governed by the law of heat conduction. It is proved that the stress will be discontinuous where the temperature is discontinuous, and the amount of the discontinuity is evaluated and found to depend only on the temperatures at the point considered. The theorem is applicable to the shrink fits of engineering practice. The fundamental problem of a heated element of volume, of arbitrary shape, in an otherwise cold infinite solid is solved and the solution represented as a centre of dilatation. Concentration of thermal stress is illustrated in a discussion of the infinite plate with elliptical and rectangular heated areas. A complete solution is given for the heating or cooling of the solid cylinder (temperatures independent of the axial coordinate).

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(1) J. M. C. Duhamel, ' Mém. . . . . par divers savants,' t. 5, p. 440.
(2) C. W. Borchardt, *Monatsberichte Akad. Wiss. Berlin*, 1873, p. 9.
(3) Carslaw, ' Conduction of Heat.'
(4) Gray and Mathews, ' Bessel Functions.'

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[*The Editors do not hold themselves responsible for the views expressed by their correspondents.*]

Fig. 1.

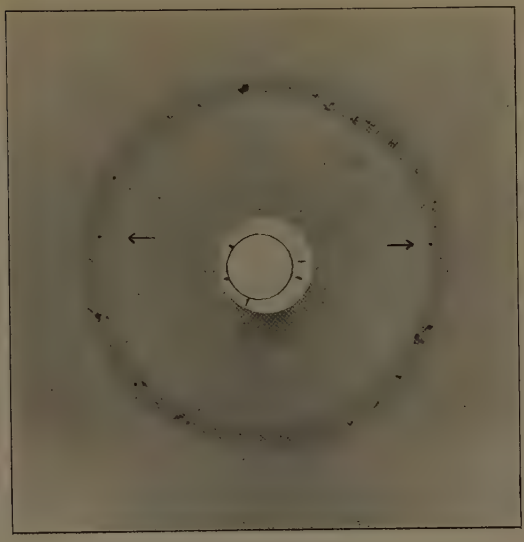
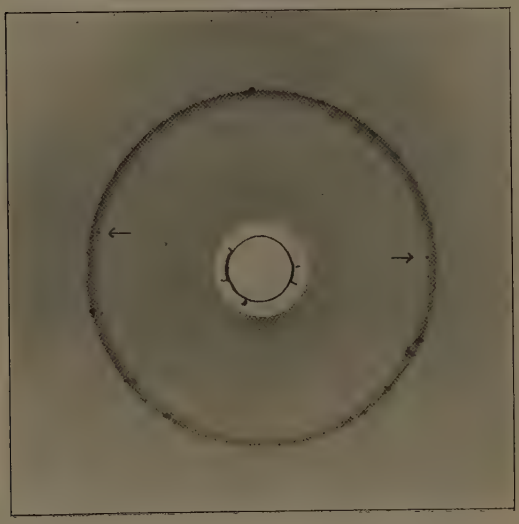


Fig. 2.



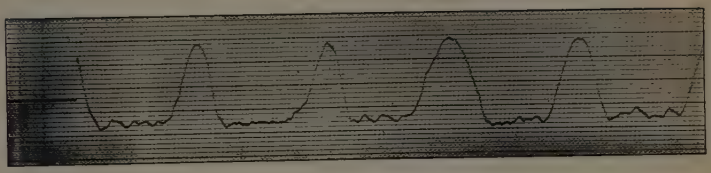
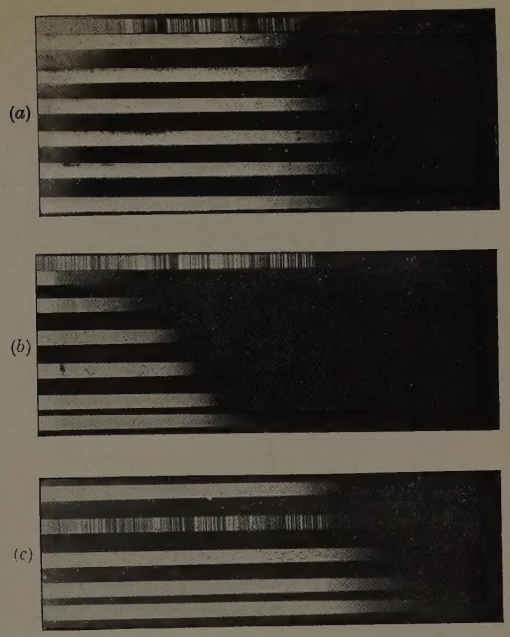
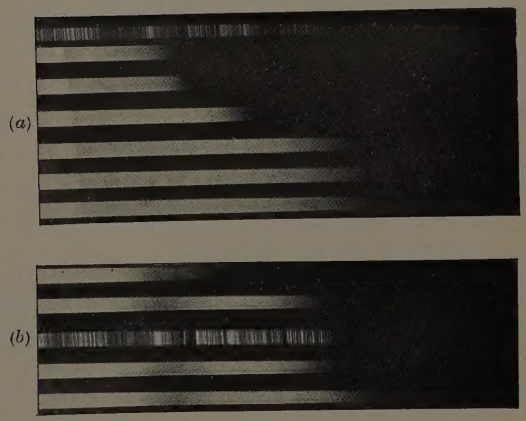


Fig. 4.



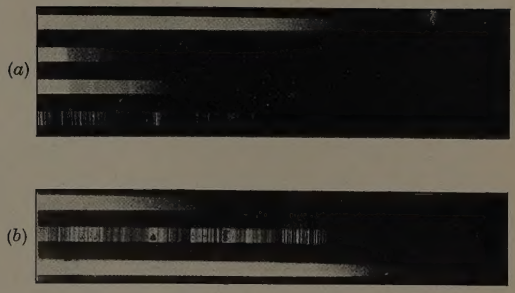
Absorption in paramagnetic chloride solutions.
(Different concentrations of salt.)

Fig. 5.



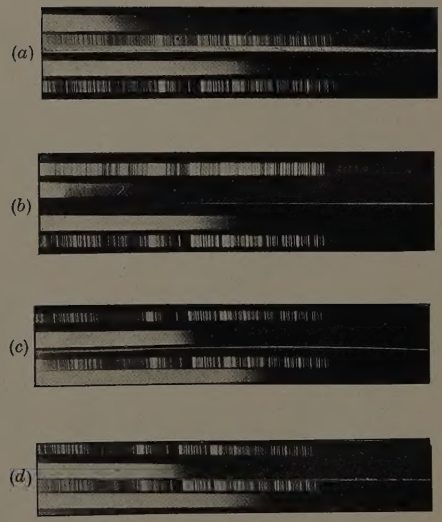
Absorption in paramagnetic chloride solutions.
(Different concentrations of HCl.)

Fig. 6.



Absorption in paramagnetic chloride solutions.
(Different solvents.)

Fig. 7.



Absorption in paramagnetic chloride solutions.
(Different temperatures.)